Field of a Point-Source of Radiation

In a Stratified Inhomogeneous Medium

By: L. M. Brekhovskikh

Izvestiya Akademii Nauk SSSR, Seriya Fizicheskaya

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I. INTEGRAL FORM OF SOLUTION

Introduction

The author (1,2), and the author in collaboration with P. A. Ryazin (3), have previously considered the propagation of sound waves and of electro-magnetic waves in strata bounded by a plane, mutually parallel, with media interfaces. We propose hereinafter to generalize the results obtained by considering wave propagation in a stratum bounded not only by "sharp" but also by "eroded" (for an elucidation of terms see Section 1), boundary surfaces.

We hope that this generalization will be found useful in connection with such problems as evaluation of the influence of the ionosphere upon propagation of radio waves, and also in the study of sound propagation in the atmosphere and in the sea. In the latter case the eroded boundaries may be the bottom of the sea in the "strata of a sudden change" (regions of large vertical gradients of temperature and salinity).

The problem of propagation of electromagnetic waves in plane-stratified media has been considered by G. A. Grinberg (4). He has pointed out a method whereby the potential of the electromagnetic field can be presented in the form of quadratures.

The same problem has been studied by P. E. Krasnushkin. In his paper (5) he has pointed out a number of interesting conditions which obtain in the course of propagation of the waves in a plane-stratified medium, and more particularly the fact that the field in such a case displays partially or fully the nature of a waveguide, i.e., constitutes a discrete set of waves, undergoing propagation at different velocities. However, the discussion presented by P. E. Krasnushkin is also of a very general nature. Derivation of quantitative results is most arduous except in a number of idealized instances. Moreover, that portion of the field which does not constitute a discrete set of waves has been entirely omitted from his considerations.

We conduct the study of the field of a point source of radiation in a plane-stratified medium under very general assumptions relative to the nature of the stratified medium. Thereby the final results can be presented in a form adapted for use in practical computations, as well as in a qualitative analysis of the field structure. In the first part of the paper we present the solution of the problem in the form of quadratures. In the second part, this solution is studied in detail using the results of our previous work (6) concerned with wave reflection from plane-stratified media.

Section 1. Statement of the Problem

Consider a stratified-inhomogeneous medium, the parameters of which depend on one rectangular coordinate z. In the electromagnetic case, the parameter of the medium is the dielectric constant ϵ (z) (we are considering those cases wherein the magnetic permeability can be assumed to be equal to one); in the acoustic case the parameters will be the density of the medium ρ (z) and sound velocity c(z). We assume that the medium satisfies the following prerequisites:

(a) When $z = \pm \infty$, the parameters of the medium acquire constant values equal to $\varepsilon_1, \rho_1, c_1$ when $z = -\infty$, and $\varepsilon_2, \rho_2, c_2$ when $z = +\infty$, irrespective of whether the parameters approach these values asymptotically or become exactly equal to them starting from some values of z.

(b) There is present a stratum bounded by planes $z = z_1$ and $z = z_2$, the medium enclosed therein being considered as homogeneous; within this stratum are enclosed the source of radiation (coordinate z_0) and the receiver (coordinate z). Thickness of the homogeneous stratum we denote by h_0 ($h_0 = z_2 - z_1$), the values of the parameters within it by $\varepsilon_0, \rho_0, c_0$.

Properties of the medium at an arbitrary point of space shall be characterized also by the index of refraction n with respect to the medium within the homogeneous stratum $z_1 < z < z_2$. In the electromagnetic case $n(z) = \sqrt{\frac{\varepsilon(z)}{\varepsilon_0}}$, and correspondingly $n_1 = \sqrt{\frac{\varepsilon_1}{\varepsilon_0}}$, $n_2 = \sqrt{\frac{\varepsilon_2}{\varepsilon_0}}$. In the acoustic case, $n(z) = \frac{c_0}{c(z)}$ and, consequently, $n_1 = \frac{c_0}{c_1}$, $n_2 = \frac{c_0}{c_2}$.

Under the conditions indicated above we have a problem pertaining to propagation of waves in a homogeneous stratum $z_1 < z < z_2$ bounded below and above by two stratified inhomogeneous half-spaces. The latter were designated hereinbefore as "eroded" boundaries.

We will use a cylindrical system of coordinates r, z, ϕ (Figure 1) (Figure 1 represents the picture within plane $\phi = 0$).

At point O on axis z is located the source of radiation, at point P (r, z, O), the receiver. We will assume the source to be a vertical dipole in the electromagnetic case, and a pulsating sphere of infinitely small radius in the acoustic case. In both cases the field of radiation can be defined by a single scalar function ψ (r, z) which with an accuracy up to a constant factor is the vertical component of Hertz vector in the electromagnetic case, and the acoustic potential in the acoustic case. On convergence of the receiver towards the source ($R \rightarrow 0$, Figure 1), function ψ must degenerated into a spheric wave $\frac{e^{ik_0R}}{R}$, wherein k_0 is the wave number for the homogeneous stratum. We are omitting throughout the factor $e^{-i\omega t}$.

Figure 1. Diagram of location of source of radiation and receiver in a cylindrical system of coordinates (r, z, φ) in the case of the problem involving propagation of waves within a homogeneous layer bounded by planes $z = z_1$ and $z = z_2$. In the drawing the origin of coordinates is located on the lower boundary of the layer so that

$$z_1 = 0$$

Section 2. Integral Form of the Solution

Let us utilize the well-known resolution of a wave, emitted from a point source, into plane waves. Using rectangular coordinates $x = r \cos \varphi$, $y = r \sin \varphi$, z, this representation can be written in the form (see for example /7/, page 138, and also /8/. The expression (1) differs from those found in the literature solely in the use of angles of slide α, in lieu of incident angles $\gamma = \frac{\pi}{2} - \alpha$.)

$$\frac{e^{ik_0 R}}{R} = -\frac{ik_0}{2\pi}\int_{\Gamma}\int_{0}^{2\pi} e^{ik_0[x\cos\alpha\cos\phi' + y\cos\alpha\sin\phi' \pm (z-z_0)\sin\alpha]} \qquad (1)$$

The expression within the sign of integration represents a plane wave, the normal to the front of which has the direction cosines $\cos\alpha\cos\phi$, $\cos\alpha\sin\phi'$, $\sin\alpha$, α being the angle of slide formed by the normal to the front of the wave with the planes z = const. Integration with respect to ϕ' is effected from 0 to 2π, and for α -- along path Γ within the complex plane (Figure 2). The necessity of resorting to complex angles in the integration is due to the impossibility of obtaining a wave with the required characteristics when R = 0 by means of superposing plane waves having only real direction cosines.

Figure 2. Integration paths and in the complex plane $\alpha = \alpha_1 + i\alpha_2$

In the exponent within the integration sign, the sign "+" is used when $z > z_0$, or the sign "-" when $z < z_0$. This is consonant with the fact that when $z > z_0$ there are present only plane waves which propagate in the direction of the positive z, whereas when $z < z_0$ the waves are propagated only in the direction of negative z.

Utilizing the equation $x\cos\phi + y\sin\phi' = r\overbrace{\gamma}^{\cos(\phi'-\phi)}$ and integral form of the Bessel function, (1) can be written as:

$$\frac{e^{ik_0R}}{R} = -ik_0 \int_{T'} e^{\pm ik_0(z-z_0)\sin\alpha} J_0(k_0 r\cos\alpha)\cos\alpha\, d\alpha \qquad (2)$$

Herein the path of integration T', extending from $\frac{\pi}{2}$ to $i\infty$ can be transformed into a path of integration T' extending from $\pi - i\infty$ through point $\alpha = \frac{\pi}{2}$, to $i\infty$ (Figure 2), the Bessel function within the sign of integration being replaced by Hankel function. This transformation is effected, for example in /7/ (page 123), with the sole difference that in the reference cited α is replaced by the variable $\xi = k_0\cos\alpha$ with the limits of integration $\pm\infty$.

As a result we have in addition to (1) and (2) a third expression for the spherical wave:

$$\frac{e^{ik_0R}}{R} = -\frac{ik_0}{2} \int_{T_1} e^{ik_0(z-z_0)\sin\alpha} H_0^{(1)}(k_0 r\cos\alpha)\cos\alpha\, d\alpha \qquad (3)$$

Weyl, in his well-known paper /8/, calculates in the following manner the field of a vertical dipole located above the partition boundary of two media. Hertz vector, or more precisely its vertical component, is given at any point by the sum of the Hertz vector of direct wave /one of the expressions (1) -- (3)/ and the Hertz vector of the reflected wave. The phase of the direct wave at point (x, y, z) is (see (1)/:

and, as can be readily ascertained, the phase of the reflected wave is,

$$k_0[x\cos\alpha\cos\phi + y\cos\alpha\sin\phi \pm (z-z_0)\sin\alpha]$$

Here $z = z_0$ is preceded by only a single sign, since reflected waves always undergo propagation in the direction of positive z.

A complete expression for the Hertz vector ψ is obtained on multiplying the amplitude of the reflected wave by the coefficient of reflection, adding the product so obtained to the direct wave and integrating the sum along all direction cosines of plane waves. We have as a result

$$\psi = -\frac{ik_0}{2\pi}\int_{\pi}\int_{0}^{2\pi} e^{ik_0(x\cos\alpha\cos\varphi + y\cos\alpha\sin\varphi)} \times$$
$$\times \left[e^{\pm ik_0(z-z_0)\sin\alpha} + V(\alpha)^{ik_0(z+z_0)\sin\alpha}\right]\cos\alpha\, d\alpha\, d\varphi \quad (4)$$

whereing $V(\alpha)$ is the reflection coefficient of Fresnel /see (11) below/.

On generalizing the considerations which lead to formula (4), an expression can be obtained for the Hertz vector and the acoustic potential in the case of the source of radiation located within the stratum.

In this case at the point of reception we have in addition to the direct wave, an infinite series of waves with a different number of reflections from the boundaries of the stratum (as this occurs, for example, in the case of a candle placed between two mirrors). The phase of each of these waves at the point of reception is given by the expression:

$$K_0(x\cos\alpha\cos\varphi' + y\cos\alpha\sin\varphi + \Delta\sin\alpha)$$

where $\Delta = z_0 + z$ for the wave reflected once from the lower boundary, $\Delta = 2h_0 - z - z_0$ for the wave reflected once from the upper boundary, $\Delta = 2h_0 + z - z_0$ for the wave reflected first from the upper, and then from the lower boundary of the partition, and so forth.

Figure 3. Diagrammatic drawing for the phase determination of the wave in the case of a varying number of reflections from the boundaries of the layer. Here are depicted four cases corresponding to the waves with the least number of reflections from the boundaries

For the determination of Δ for a different number of reflections from the boundaries, it is convenient to utilize the diagrammatic drawing shown in Figure 3, for the direct wave, and the first three waves having the least number of reflections. Herein the value Δ is equal to the sum of the vertical portions of the broken lines joining points O and O'. (The location of points O' and O in figures 3 and 4 /see below/ is chosen for convenience of diagramming and does not coincide with the actual location.

The total field is obtained on summation of all plane waves having the same direction cosines but differing in the number of reflections from the boundaries, and integrating this sum along all direction cosines. The coefficients of reflection of plane waves from lower and upper boundaries are denoted respectively by

$V_1 = V_1(\alpha)$ and $V_2 = V_2(\alpha)$. In the determination of, for example, $V_1(\alpha)$, one should visualize the space $z < 0$ as being supplemented above by a homogeneous half-space having the same properties as those of the medium within the stratum, and that in this half-space is present the given incident plane wave. As a result, on introducing the denotation

$$b = ik_0 \sin\alpha \tag{5}$$

we have the complete expression for the Hertz vector on the acoustic potential ψ :

$$\psi = -\frac{ik_0}{2\pi}\int_{\Gamma}\int_0^{2\pi} e^{ik_0(x\cos\alpha\cos\phi' + y\cos\alpha\sin\phi')} \times$$
$$\times \sum_{l=0}^{\infty}\Big[e^{b(z_0-z)} + V_1 e^{b(z_0+z)} + V_2 e^{b(2h_0-z-z_0)} + \tag{6}$$
$$+ V_1V_2 e^{b(2h_0+z-z_0)}\Big] \times (V_1V_2)^l e^{2lbh_0} \cos\alpha \, d\alpha \, d\phi',$$

wherein for the sake of definiteness we have taken the case $z < z_0$.

The four case shown in Figure 3 correspond here to the term of the sum with $l = 0$.

In the last expression, the same as in passing from expression (1) to expression (3), integration can be effected with respect to ϕ and the integral along path Γ be transformed into an integral along Γ_1 . If in addition we take into account the equality

$$e^{b(z_0-z)} + V_1 e^{b(z_0+z)} + V_2 e^{b(2h_0-z-z_0)} + V_1V_2 e^{b(2h_0+z-z_0)} \equiv$$

$$\equiv (e^{-bz} + V_1 e^{bz}) \left(e^{-b(h_0 - z_0)} + V_2 e^{b(2h_0 - z_0)} \right)$$

and the value of the sum of series:

$$\sum_{l=0}^{\infty} (V_1 V_2)^l e^{2l \cdot b h_0} = \frac{1}{1 - V_1 V_2 e^{2bh_0}}$$

we have:

$$\psi = -\frac{ik_0}{2} \int_{\Gamma_1} \frac{(e^{-bz} + V_1 e^{bz})(e^{-b(h_0 - z_0)} + V_2 e^{b(h_0 - z_0)})}{e^{-bh_0}(1 - V_1 V_2 e^{2bh_0})} \times \tag{7}$$

$$\times H_0^{(1)}(K_0 r \cos\alpha) \cos\alpha \, d\alpha.$$

In the case when $z > z_0$, ψ is obtained from (7) by inter changing z and z_0.

Thus the Hertz vector or the acoustic potential is found to be given in the form of an integral, with the function within the sign of integration containing the reflection coefficients of plane waves from boundaries of the stratum. These coefficients are well known in many instances (for example in the case of partition boundary of two homogeneous media, for a plate of finite thickness, and so forth), or can be calculated.

In those instances when the exact expression of the reflection coefficient cannot be obtained, use can be made of the results of our prior work /6/, whereing the reflection coeeficient in an arbitrary case is determined by means of the method of successive approximations.

It is not difficult to see that the expression (7) satisfies the wave equation $\Delta\psi + k_0^2\psi = 0$. The proof that it also satisfies boundary conditions and exhibits the necessary characteristic when R = 0, we give in the supplement. There also is shown that the integral (7) converges.

In the seledtion of stratum boundaries z_1 and z_2 we are limited only by the condition that the medium within this stratum must be sufficiently homogeneous. Otherwise the selection of boundaries is arbitrary. This however does not lead to any indefiniteness in the value of ψ .

(Footnote: We are considering the stratum $z_1 < z < z_2$ as being sufficiently homogeneous if on passage through the stratum of a plane wave, the amplitude of the reflected wave generated in the course of passage will be sufficiently small in comparison with amplitudes of waves reflected from the half-spaces bounding the stratum. Phase advance must also not differ substantially from phase advance in a homogeneous space. In such a case on displacement, for example, of the upper boundary downwards by δ , $V_2(\alpha)$ will change, and the alteration of its value differs from the initial value by the additional phase factor $e^{-2ik_0\delta\sin\alpha}$. This follows directly from the definition of $V_2(\alpha)$ as the ratio of the Hertz vector, or of acoustic potential, at a given point z, in the reflected and the incident waves. For details on characteristics of $V(\alpha)$, see /6/. On such displacement of the boundary h_0 decreases by δ . It can be readily ascertained that as a result the expression (7) for ψ remains the same.)

If the source and receiver are located along the same horizontal, then on selecting as the origin of the coordinate system the point of location of the source of radiation, we have $z = z_0 = 0$; Thickness of the stratum can also be selected as bing equal to zero. As a result we have from (7):

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$$\psi = -\frac{ik_0}{2}\int_{T_1}\frac{(1+V_1)(1+V_2)}{1-V_1V_2}\times$$
$$\times H_0^{(1)}(k_0 r\cos\alpha)\cos\alpha\, d\alpha. \tag{8}$$

This formula can also be used in those cases of plane-stratified media when a homogeneous stratum of finite thickness cannot be segregated. In such instances formula (8) can be substantiated in the following manner. The true dependence of the parameters of the medium upon the coordinate z, we replace mentally, by a certain imaginary dependence, approximating as much possible the true dependence, but also permitting segregation of a homogeneous layer having a thickness h_o. The field in this artificially created homogeneous stratum will be set forth by formula (7). Effecting thereafter the limit transition $h_o \longrightarrow 0$, we combine the imaginary dependence with the actual, and in so doing formula (7) becomes formula (8). At the same time V_1 in (8) must be considered as being the coefficient of reflection from the half-space $z < 0$, visualizing this half-space as being supplemented by a homogeneous half-space disposed above it and having the same parameters as the medium with $z = 0$.

Section 3. Individual Forms of Solution

Let us consider the expression (7) for a number of individual cases.

(a) When $V_1 = V_2 = 0$, we obtain from (7), as expected, the expression (3) for the Hertz vector or the acoustic potential of the sources of radiation located in a limitless homogeneous space.

(b) Let $V_1 \neq 0$, $V_2 = 0$. Then from (7) and an analogous formula for $z > z_0$ we have:

$$\psi = -\frac{ik_0}{2}\int_{\Gamma_1}\left[e^{\pm b(z-z_0)} + V_1 e^{b(z+z_0)}\right] \times \times H_0^{(1)}(k_0 r\cos\alpha)\cos\alpha\, d\alpha \qquad (9)$$

(the sign "+" being used when $z > z_0$, and the sign "-" when $z < z_0$)1 In this instance the upper boundary is absent, and the source of radiation and receiver are both within a homogeneous half-space.

Formula (9) will be obtained, of course, also from (4) if this is integrated with respect to φ' and the integral along Γ is transformed into an integral along Γ'.

On the basis of expression (9) problems can be solvedwhich are analogous to the problem of Sommerfeld but differing from it in that the source of radiation and receiver are located above the boundary underneath which the medium can display an arbitrary stratified nature. Thus, the extimate of the effect of a snow cover on propagation of radio waves along the earth's surface, by using formula (9) is reduced to an estimate of its effect upon the reflections coefficient of plane waves.

(c) Let the lower half-space be homogeneous the same as the upper one, and be separated therefrom by a sharp partition boundary. Then we have in the acoustic case:

$$V_1(\alpha) = \frac{\rho_1 \sin\alpha - \rho_0\sqrt{n_1^2 - \cos^2\alpha}}{\rho_1 \sin\alpha + \rho_0\sqrt{n_1^2 - \cos^2\alpha}} \qquad (10)$$

and in the electromagnetic case (see /8/ page 80)

$$V_1(\alpha) = \frac{n_1^2 \sin\alpha - \sqrt{n_1^2 - \cos^2\alpha}}{n_1^2 \sin\alpha + \sqrt{n_1^2 - \cos^2\alpha}} \tag{11}$$

If the source of radiation is located on the partition boundary, then assuming in (9) $z_0 = 0$, and taking into account (11), we have in the electromagnetic case:

$$\psi = -ik_0 n_1^2 \int_{\Gamma_1} \frac{e^{ik_0 z \sin\alpha} H_0^{(1)}(k_0 r \cos\alpha) \cos\alpha \sin\alpha \, d\alpha}{n_1^2 \sin\alpha + \sqrt{n_1^2 - \cos^2\alpha}} \tag{12}$$

Taking into account that $n_1 = \frac{k_1}{k_0}$, and passing to a new integration variable $\xi = k_0$, we obtain from (12) the well-known formula of Sommerfeld (see /7/, page 123):

$$\psi = k_1^2 \int_{-\infty}^{+\infty} \frac{e^{-z\sqrt{\xi^2 - k_0^2}} H_0^{(1)}(\xi r)\, \xi \, d\xi}{k_0^2\sqrt{\xi^2 - k_1^2} + k_1^2\sqrt{\xi^2 - k_0^2}} .$$

(d) If we take for $V_2(\alpha)$ expressions analogous to (10) and (11), when we have from (7) the previously considered instance of wave propagation in a stratum bounded by two homogeneous half-spaces.

<u>Supplement</u>

Let us prove that the expression (7) for Hertz vector or the acoustic potential satisfies the corresponding boundary conditions and possesses the required characteristic at the point $z = z_0$, $r = 0$.

Consider the boundary conditions at the lower boundary, assuming $z_1 = 0$. First of all, we obtain the expression for the Hertz vector with $z < 0$, denoting it by ψ_1. In so doing, we proceed in the same manner as for ψ, i.e., we resolve the spherical wave into plane waves and summate the successive reflections of each of the plane waves. Figure 4 shows diagrammatically the four simplest ways whereby the plane wave can reach from the source of radiation an arbitrary point O' within the lower medium. Let us consider first one of the plane waves into which is resolved a spherical wave (See /1/). Let the plane wave, on its direct passage from) to O' (Figure 4, a), form at O' the field,

$$\exp\left[ik_0(-x\cos\alpha\cos\phi' + y\cos\alpha\sin\phi' + z_0\sin\alpha)\right] f(z,\alpha).$$

We are not concerned with the form of function f, which determines the nature of the inhomogeneous medium when $z < 0$. The field formed by the wave on a single reflection from the upper boundary (Figure 4, b) will be obviously:

$$\exp\left\{ik_0\left[-x\cos\alpha\cos\phi' + y\cos\alpha\sin\phi' + (2h_0 - z_0)\sin\alpha\right]\right\} V_2(\alpha) f(z,\alpha),$$

Since the entire difference from that of the first case consists solely in the fact that the plane wave has previously travelled over a longer path within the stratum and has been reflected once from the upper boundary.

Figure 4. Diagrammatic drawing for the determination of phases of individual waves constituting the total field in the lower half-space. The same as in Figure 3, four cases are represented here corresponding to waves with the least number of reflections from the boundaries of the stratum: (a) direct passage of plane wave from point O to point O'; (b) one reflection from upper boundary on passage from O to O'; (c) one reflection from upper boundary; (d) two reflections from upper boundary and one from the lower.

In an analogous manner are obtained the fields formed by waves having had a greater number of reflections (See Figure 4, c and d). On summating them and integrating over angles α and ϕ', we have:

$$\psi_1 = \frac{ik_0}{2\pi} \int_{\pi} \int_0^{2\pi} e^{ik_0(x\cos\alpha\cos\phi' + y\cos\alpha\sin\phi')} \times$$

$$\times \sum_{\ell=0}^{\infty} \left[e^{bz_0} + V_2^* e^{b(2h_0 - z_0)} \right] (V_1 V_2)^{\ell} e^{-2\ell b h_0} \times$$

$$\times f(z, \alpha) \cos\alpha \, d\alpha \, d\phi'$$

The four waves represented diagrammatically in Figure 4 correspond to instances when $\ell = 0$ and $\ell = 1$.

In the same manner as in the case of ψ (See Section 2), the last given expression is reduced to the form:

$$\psi = -\frac{ik_0}{2}\int_{\Gamma_1} \frac{e^{-b(h_0-z_0)} + V_2 e^{b(-h_0-z_0)}}{e^{-bh_0}\left(1 - V_1V_2 e^{2bh_0}\right)} \times$$

$$\times f(z,\alpha)\, H_0^{(1)}(k_0 r\cos\alpha)\cos\alpha\, d\alpha \qquad \text{(I)}$$

Boundary conditions require that with z = 0

$$\frac{\partial\psi}{\partial z} = \frac{\partial\psi_1}{\partial z}, \quad \psi = \frac{\rho'}{\rho_0}\psi_1 , \qquad \text{(II)}$$

In the electromagnetic case $\frac{\rho'}{\rho_0}$ must be replaced by In this instance ($\frac{K'}{K_0}$ and k' are the density and wave number in the lower medium in the immediate vicinity of the boundary. If the properties of the medium or passage through the boundary change continuously, then $\frac{\rho'}{\rho} = 1$, $\frac{K'}{K_0} = 1$,

Substituting the expressions (7) and (I) in (II) and equalizing the expressions within the sign of integration we obtain two equations:

$$ik_0 \sin\alpha\left[V_1(\alpha) - 1\right] = \left(\frac{\partial f}{\partial z}\right)_{z=0}$$

and

$$1 + V_1(\alpha) = \frac{\rho'}{\rho_0} f(0,\alpha).$$

The same equations are obtained on considering the problem of reflection of a plane wave. They are naturally satisfied if the proper value is given to the reflection coefficiants $V_1(\ \ )$.

Let us now demonstrate that the integral (7) converges everywhere except at point $z = z_0$, $r = 0$, where it has the same characteristic as that of the spheric wave (3). In doing this we must study $V_1(\alpha)$ and $V_2(\alpha)$ within the complex plane $\alpha = \alpha_1 + i\alpha_2$, for which purpose we will utilize the results of our prior work /6/. It was shown there that the reflection coefficient $V_1(\alpha)$ can be written in the following form:

$$V_1(\alpha) = \frac{\frac{\rho_1}{\rho_0}\sin\alpha\, u(z_1) - \sqrt{n_1^2 - \cos^2\alpha}}{\frac{\rho_1}{\rho_0}\sin\alpha\, u(z_1) + \sqrt{n_1^2 - \cos^2\alpha}} \qquad \text{(III)}$$

where function u(z) is determined from the equation:

$$\frac{du}{dz} = ik_0 \frac{\rho(z)}{\rho_1}\sqrt{n_1^2 - \cos^2\alpha}\left(1 - \frac{\rho_1^2}{\rho^2(z)} \times \frac{n^2(z) - \cos^2\alpha}{n_1^2 - \cos^2\alpha}\right) \qquad \text{(IV)}$$

for boundary condition $z \to -\infty$, $u \to 1$.

In the case of complex $\sqrt{n_1^2 - \cos^2\alpha}$, the sign of the root must be chosen from the condition (V) below. (Here, unlike in /6/, the angle of slide of the incident wave was foundto be more conveniently denoted by α, and not by α_0. Moreover, since the time factor is used here in the form $e^{-i\omega t}$ and not $e^{i\omega t}$, in contradistinction with /6/ + i appears in all the formulas in lieu of --i.) This is necessary in order to have with $z \to -\infty$, a damping wave (Imx denotes the imaginary portion of x).

$$\operatorname{Im}\sqrt{n_1^2 - \cos^2\alpha} > 0 \qquad \text{(V)}$$

In the electromagnetic case $\frac{\rho_1}{\rho_0}$ cannot be replaced by n_1^2, and $\frac{\rho(z)}{\rho_1}$ by $\frac{n^2(z)}{n_1^2}$.

On remote sectors of the path of integration (Figure 2), taking into account that $\cos\alpha = \cos\alpha_1 \operatorname{ch}\alpha_2 - i\sin\alpha_1 \operatorname{sh}\alpha_2$ and $\sin\alpha = \sin\alpha_1 \operatorname{ch}\alpha_2 + i\cos\alpha_1 \operatorname{sh}\alpha_2$, we will have $|\cos\alpha| \approx |\sin\alpha| \to \infty$. In addition, taking into account the condition (V) we have $\sqrt{n_1^2 - \cos^2\alpha} \to \sin\alpha$. As a result, equations (III) and (IV) assume the following limit form:

$$V_1 = \frac{\frac{\rho_1}{\rho_0}u(z_1) - 1}{\frac{\rho_1}{\rho_0}u(z_1) + 1} \tag{III'}$$

and

$$\frac{1}{ik_0 \sin\alpha} \cdot \frac{du}{dz} = \frac{\rho(z)}{\rho_1}\left(1 - \frac{\rho_1^2}{\rho^2(z)}u^2\right). \tag{IV'}$$

When $|\sin\alpha| \to \infty$, from the last equation we have $u(z) = \frac{\rho(z)}{\rho_1}$. This solution also satisfies the boundary condition. Further, on assuming $z = z_1$, we have $u(z_1) = \frac{\rho_0}{\rho_1}$, and in accordance with (III'), $V_1 = 0$.

In some cases it is necessary to know the nature of the tendency of V_1 toward zero. For these cases, utilizing the result obtained for $u(z)$ as a first approximation and substituting it into the left hand portion of (IV'), we have the second approximation for $u(z)$. On substituting it into (IV') we have

$$V_1 = \frac{i}{4k_0 \sin\alpha}\left(\frac{1}{\rho} \cdot \frac{d\rho}{dz}\right)_{z_1} \tag{VI}$$

The results obtained relate to the case wherein the density of the medium (refraction index in the electromagnetic instance) is changing continuously. If, however, the change seems to occur suddenly, then formula (10) gives, on limiting transition $|\alpha| \to \infty$

$$V_1(\alpha) = \frac{\frac{\rho_1}{\rho_0} - 1}{\frac{\rho_1}{\rho_0} + 1},$$

where $\frac{\rho_1}{\rho_0}$ is the ratio of densities on either side of the boundary.

Thus in all instances $V_1(\alpha)$ tends toward zero or a finite value. An analogous result takes place for $V_2(\alpha)$. Taking this into account, as well as the fact that the real portion of b is negative, the expression within the sign of integration in (7) can be written, for remote portions of path Γ_1, in the following form (with $z < z_0$):

$$e^{ik_0(z_0-z)\sin\alpha} H_0^{(1)}(k_0 r\cos\alpha)\cos\alpha\, d\alpha,$$

which coincides with the function within the integration sign in expression (3) of the spherical wave. Using the asymptotic representation of the Hankel function and introducing a new variable $\xi = k_0\cos\alpha$, integration with respect to which is effected along the real axis from $-\infty$ to $+\infty$, we have, with an accuracy within the constant factor:

$$e^{-\sqrt{\xi^2 - k_0^2}(z_0-z) + i\xi r}\frac{d\xi}{\sqrt{\xi}}$$

It is apparent therefrom that when the limits of integration tend towards $\pm\infty$, we will always have a converging integral with the exception of point $z = z_0$, $r = 0$. At this point the integral will diverge in the same way as the integral (3) of a spherical wave.

LITERATURE CITED

1. Brekhovskikh, L. M., DAN (Doklady Akademii Nauk) 48, 422 (1945).

2. Brekhovskikh, L. M. Izvestiya Akademii Nauk SSSR, Seriya Fizicheskaya, 10, 491, 1946.

3. Ryazin, P. A., and Brekhovskih, L. M., Izvestiya Akademii Nauk SSSR, Seriya Fizicheskaya, 10, 285, 1946.

4. Grinberg, G. A., DAN, 26, 581, 1946.

5. Krasnushkin, P. Ye., Metod normal'nykh voln v primenenii k probleme dal'nykh radiosvyazey /Normal waves method in its application to distant radio communications/, MGU, 1947; ZhTF, 18, 431, 1948.

6. Brekhovskihk, L. M., ZhTF 19, 10, 1126, 1949.

7. Vvedenskiy B. A., Osnovyteorii rasprostraneniya radiovohn /Fundamentals of the theory of radio waves propagation/, GTTI, 1934.

8. Weyl, H., Ann. der Physik, 60, 481, 1919.

END

II. DISCUSSION OF THE SOLUTION

In the first part of the paper /1/ the author has proposed a new method for the solution of the problem pertaining to the field of a point source of radiation in a stratified-inhomogeneous medium. In this method the spherical wave radiated by the point source is resolved into plane waves, after which the behavior of each of these plane waves is analyzed separately. It is not difficult to see the very close analogy between this method and methods of solving diffraction problems. In the latter instance the incident wave is usually resolved into waves having the same symmetry as that of the body upon which diffraction takes place. ~~See~~ In the problem under consideration herein the plane waves fully correspond, from the standpoint of symmetry, to the plane stratified medium within which the field is being studied.

In the first part we have obtained the solution in an integral form. In the present, second part, we will make an analysis of the integrals obtained and will show that the method proposed by us makes it possible substantially to advance in the solutions of the problem and to present results in a form adapted to computation. In hitherto published papers, as pointed out in /1/, the authors have confined themselves to a presentation of results in a general form, completing the computations only for a number of highly idealized specific instances.

Section 1. Study of the Basic Integral

Resolution of a spherical wave into plane waves includes, as is known (See /1/), in addition to the usual plane waves, the so-called inhomogeneous waves. The latter permit the same formal mathematical description as the usual plane waves, and only the inclination angles must be considered as being complex. As a result, the spherical wave is represented in the form of an integral by plane waves in which integration is effected within the complex plane of the angles. An analogous integral represents also the total field within the plane-stratified medium. In the first part of the paper we have obtained such an expression for the acoustic potential in the case of acoustics, and for the vertical component of the Hertz vector, in electrodynamics (Formula (7)):

$$\psi = \int_{l_1} \Phi(\alpha) H_0^{(1)}(k_0 r \cos\alpha) \cos\alpha \, d\alpha, \qquad (1)$$

wherein

$$\Phi(\alpha) = \frac{ik_0}{2} \cdot \frac{(e^{-bz} + V_1 e^{bz})(e^{-b(h_0 - z_0)} + V_2 e^{b(h_0 - z_0)})}{e^{-bh_0}(1 - V_1 V_2 e^{2bh_0})} \qquad (2)$$

It is here assumed that source of radiation and receiver are located within a homogeneous stratum of thickness h_0 bounded below and above by arbitrarily stratified media.

In expression (2) the following denotations are used: $V_1 = V_1(\alpha)$ and $V_2 = V_2(\alpha)$ are coefficients of reflection of plane waves as a function of the angle of sliding α, from lower and upper media, respectively, which media bound the stratum; z_0 and z, respectively, the distance of the source of radiation and the receiver from the lower boundary of the stratum; r is the horizontal distance between the source of radiation and the receiver; and finally

$$b = i k_0 \sin \alpha , \tag{3}$$

k_0 being the wave number in the medium which constitutes the stratum.

Expression (2) relates to the instance when $z < z_0$. Transition to the case when $z > z_0$ is effected by replacing z by z_0 and vice versa.

Path Γ_1 , along which integration is effected in (1) is located within the complex plane of angles $\alpha = \alpha_1 + i\alpha_2$, and extends from $\alpha = \pi + i\infty$ to $\alpha = \pi - i\infty$, then along the real axis to $\alpha = \pi$, and finally along the imaginary axis to $\alpha = 0$ (Figure 1). Integral along path Γ_1 can be replaced by the sum of integrals along Γ_2 and Γ_3 , use of which will be made hereinafter.

<u>Figure 1</u>. Paths of integration Γ_1 , Γ_2 , Γ_3 , and Γ_4 in the complex plane $\alpha = \alpha_1 + i\alpha_2$

Let us analyze integral (1) first for the instance when $n_1 \neq 1$, $n_2 \neq 1$, denoting by n_1 and n_2, the same as in /1/, the refraction indexes of the medium for $z = -\infty$ and $z = +\infty$, respectively. The index of refraction of the medium within the homogeneous stratum is assumed to be the equal to unity.

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In the instance under consideration, the integral along Γ_2 is identically equal to zero since the function within the integration sign in (1) is odd with respect to α. (This can be readily shown by taking into account that according to formulas (III) and (IV), in the supplement of /1/, we have $V_1(-\alpha) = \frac{1}{V_1(\alpha)}$, and analogously, $V_2(-\alpha) = \frac{1}{V_2(\alpha)}$.

As a result, there merely remains to determine the integral (1) along the path Γ_3. Let us extend path Γ_3 into the negative-imaginary infinity so that it will become an infinitely remote path Γ_4 (Figure 1). By a method analogous to that used in the supplement of /1/ it can be shown that the integral along path is equal to zero. As a result, the path of integration becomes "suspended" on specific points of the expression within the sign of integration, and the integral is transformed into the form:

$$\psi = 2\pi i \sum_{\ell} \operatorname{Res} \Phi(\alpha_1) H_0^{(1)}(k_0 r \cos\alpha_1) + \int_{\frac{\pi}{2} - i\infty}^{A_1^+} \Phi(\alpha) H_0^{(1)}(k_0 r \cos\alpha) \cos\alpha \, d\alpha + \int_{\frac{\pi}{2} - i\infty}^{A_2^+} \Phi(\alpha) H_0^{(1)}(k_0 r \cos\alpha) \cos\alpha \, d\alpha. \tag{4}$$

Here the first term constitutes a sum of substractions of the expression within the sign of integration, at poles located within the regions included on deformation of the path. In Figure 2 these poles are denoted by P_1, P_2, ... Their location is determined by roots of equation

$$1 - V_1(\alpha) V_2(\alpha) e^{2 i k_0 h_0 \sin\alpha} = 0 \tag{5}$$

The last two terms of (4) are integrals along edges of sections starting at points α = arc cos n_1 and α = arc cos n_2 (A_1 and A_2 in Figure 2) and extending respectively along lines $Im\sqrt{n_1^2-\cos^2\alpha}=0$ and $Im\sqrt{n_2^2-\cos^2\alpha}=0$. The apparition of these integrals is due to the fact that functions $V_1(\alpha)$ and $V_2(\alpha)$, and hence also function $\phi(\alpha)$, are not simple. Indeed, they compose roots $\sqrt{n_1^2-\cos^2\alpha}$, $\sqrt{n_2^2-\cos^2\alpha}$ (see (V) in Supplement of /1/), and the sign of each of them can be selected in two different ways. Thus we have four sign combinations (++, +-, -+, --), i.e., the functions within the sign of integration will have four values. It can be made to have a single value on a four-laminar Riemann surface. Integration paths T_1', T_2', and T_3', are located on that one of these laminas (we will call it "upper") wherein $Im\sqrt{n_1^2-\cos^2\alpha}>0$ and $Im\sqrt{n_2^2-\cos^2\alpha}>0$ (See formula (V) in /1/). Path T_3' will be located on the same lamina only in that instance when we construct the sections of the complex plane extending from branch points A_1 and A_2 and circuit them on deformation of path of integration T_3' into T_4'. If this is not done, then path T_4' will cross both sections and one of its ends will be located within the lamina where $Im\sqrt{n_1^2-\cos^2\alpha}<0$, $Im\sqrt{n_2^2-\cos^2\alpha}<0$. By the same token, it will not be connected with the corresponding end of T_3', located in the upper lamina. As a result of this, on taking into account all these singularities, we have the path of integration suspended upon poles and sections, as is shown in Figure 2.

Using the terminology of P. Ye. Krasnushkin /2/, the field represented by the first item of (4) can be designated a discrete spectrum, and that represented by the remainder of (4), the continuous spectrum. ~~tinuous spectrum~~

Figure 2. Picture in the complex plane α, obtained following deformation of contour Γ_3

Section 2. Lateral Waves

The last two integral terms of (4) represent lateral waves, which in the case of ordinary partition boundaries have been sdudied in detail in previous papers /3, 4/.

Let us consider, for example, the first of these two integrals, and designate it by W_1. It can be divided into two integrals, one along each edge of the section:

$$W_1 = \int_{\frac{\pi}{2} - i\infty}^{A_1} \phi(\alpha) H_0^{(1)}(k_0 r \cos\alpha) \cos\alpha \, d\alpha + \int_{A_1}^{\frac{\pi}{2} - i\infty} \phi^+(\alpha) H_0^{(1)}(k_0 r \cos\alpha) \cos\alpha \, d\alpha . \tag{6}$$

Herein $\phi(\alpha)$ and $\phi^+(\alpha)$ are values of the function at neighboring points located respectively upon the right and left edge of the section. As is known, on circuiting about a branch point the root sign acquires the opposite value (see, for instance, /5/, Section 92); hence $\phi^+(\alpha)$ is obtained from $\phi(\alpha)$ by changing the sign of this root. (In $\phi(\alpha)$, i.e., to the right of section, $Re\sqrt{n_1^2 - \cos^2\alpha} > 0$. In fact, at point $\alpha = \frac{\pi}{2}$ located also to the right of the section, $\sqrt{n_1^2 - \cos^2\alpha} = n_1$, but $Re\, n_1 > 0$.

On changing the direction of integration in the first of the two integrals to the opposite direction, we can combine both integrals into one, which coincides with the last integral of (6), except that $\Phi^{+}(\alpha)$ is replaced by $\Phi^{+}(\alpha)-\Phi(\alpha)$. Utilizing the value $\Phi(\alpha)$, and denoting by

$$\chi_1 = \arccos n_1 \tag{7}$$

we have, on performing simple calculations:

$$W_1 = \frac{ik_0}{2}\int_{\chi_1}^{\frac{\pi}{2}-i\infty} \frac{(V_1-V_1^{+})\left(V_2 e^{b(h_0-z)}+e^{-b(h_0-z)}\right)\left(V_2 e^{b(h_0-z_0)}+e^{-b(h_0-z_0)}\right)}{\left(e^{-bh_0}-V_1V_2e^{bh_0}\right)\left(e^{-bh_0}-V_1^{+}V_2e^{bh_0}\right)} \times \tag{8}$$

$$\times\, H_0^{(1)}(k_0 r\cos\alpha)\cos\alpha\, d\alpha,$$

wherein we obtain V_1^{+} from V_1 by changing the sign of the root $\sqrt{n_1^2-\cos^2\alpha}$. For further calculations, assuming k_0r to be large (exact criterion, see below, formulas (19) and (20), we assume

$$H_0^{(1)}(k_0 r\cos\alpha) \approx \frac{\sqrt{2}}{\sqrt{\pi k_0 r\cos\alpha}}\, e^{i\left(k_0 r\cos\alpha-\frac{\pi}{4}\right)} \tag{9}$$

Now let us apply to integral (8) the method of fastest descent.

To do this we introduce in lieu of α the new variable s, in accordance with the equation

$$\cos\alpha = n_1 + is^2 \tag{10}$$

and we subject to deformation the path of integration so that it extends from point A_1, for which s = 0, along a line corresponding to the real values of s. (On deformation we may encounter the poles of the expression within the sign of integration; we will defer this problem to the subsequent paragraph.) Taking the real terms of both sides of equation (10), we obtain for this path of integration within the complex plane $\alpha = \alpha_1 + i\alpha_2$, the equation:

$$\cos \alpha_1 \, \mathrm{ch}\, \alpha_2 = n_1 , \tag{11}$$

while assuming n_1 to be real (in the case when n_1 has an appreciable imaginary portion, the lateral waves are of no interest since they are found to be rapidly attenuated in space). In Figure 3, a and b, this path is shown in solid lines for $n_1 < 1$ and $n_1 > 1$, respectively. The dotted lines of these drawings show the sections.

Figure 3. Solid lines: paths of integration following deformation in the instances where n_1 1 (a) and n_1 1 (b). Dotted lines represent sections.

Substituting (10) and (9) in (8), we obtain within the sign of integration the exponent $e^{-k_o r s^2}$, with the integration being now effected for real values of s from 0 to ∞.

Since $k_o r$ is assumed to be large, the rapid decrease of the above-mentioned exponent with increase of s, renders material within the sign of integration, only small values of s, not in excess of the order of

$$S_{max} = \frac{1}{\sqrt{k_0 r}} \tag{12}$$

Therefore, for an approximate determination of integral (8), the expression within the sign of integration, with the exception of exponent $e^{-k_0 r s^2}$ and the difference $V_1 - V_1^+$, CAN BE TAKEN OUTSIDE the integration sign for the value of s = 0. The difference $V_1 - V_1^+$ when s =) becomes zero, since V_1 and V_{1+}^+ differ from each other only in the sign of root $\sqrt{n_1^2 - \cos^2\alpha}$, which is equal to zero when s = 0. By means of (10) it is not difficult to obtain

$$\sqrt{n_1^2 - \cos^2\alpha} \approx \sqrt{2n_1} \cdot e^{-i\frac{\pi}{4}} \cdot s.$$

We postulate

$$V_1 - V_1^+ = 2B\sqrt{n_1^2 - \cos^2\alpha} = \\ = 2B_1\sqrt{2n_1} \cdot e^{-\frac{i\pi}{4}} s. \tag{13}$$

which corresponds to expanding this difference in a series by power of s and limiting it to the first power. Herein B_1 is a constant quantity; the method of calculating the same will be indicated hereinafter.

Taking further into account that

$$d\alpha = \frac{2s\,ds}{i\sqrt{1-m_1^2-2im_1s^2+s^4}} \approx \frac{2s\,ds}{i\sin\chi_1} \tag{14}$$

and that

$$\int_0^\infty e^{-k_0 r s^2} s^2\,ds = \frac{1}{4k_0 r}\sqrt{\frac{\pi}{k_0 r}}\,. \tag{15}$$

We obtain from (8)

$$W_1 = \frac{iB_1 m_1}{k_0 r^2 \sin\chi_1} \times$$
$$\times \frac{\left[V_2(\chi_1)e^{\varkappa_1(h_0-z)} + e^{-\varkappa_1(h_0-z)}\right]\left[V_2(\chi_1)e^{\varkappa_1(h_0-z_0)} + e^{-\varkappa_1(h_0-z_0)}\right]}{\left[e^{-\varkappa_1 h_0} - V_1(\chi_1)V_2(\chi_1)e^{\varkappa_1 h_0}\right]^2} \times \tag{16}$$
$$\times\, e^{ik_0 m_1 r}\,,$$

wherin

$$\varkappa_1 = k_0\sqrt{m_1^2-1} = ik_0\sin\chi_1\,. \tag{17}$$

The expression (16) does not change on replacing z by z_0 and vice versa, and consequently it is usable for $z < z_0$ as well as for $z < z_0$.

The lateral wave represented by expression (16), the same as in the case of a stratum bounded by sharp partition boundaries, undergoes propagation along the stratum with a velocity equal to the propagation velocity within the lower medium, at a sufficiently large distance from the stratum boundary where the medium can already be considered as being homogeneous, and possesses an amplitude which decreases with distance as $\frac{1}{r^2}$. Dependency of amplitude upon z is more complex and displays a materially different focus for $n_1 > 1$ and $n_1 < 1$. In particular for $n_1 > 1$ and sufficiently large, we have, on disregarding the second terms within brackets in (16)

$$W_1 = \frac{i B_1 n_1}{K_0 r^2 \sin^2 \chi_1} \cdot \frac{e^{\varkappa_1 (z + z_0)}}{V_1^2(\chi_1)} e^{i K_0 n_1 r} \qquad (18)$$

In this instance the amplitude of the lateral wave decreases exponentially with increasing distance from lower boundary of the stratum (increasing z), i.e., the wave appears to spread along the lower boundary.

In an analogous manner, the integral along edges of second section will yield a lateral wave W_2, which undergoes propagation along the stratum at a velocity equal to the velocity within the upper medium. This analytic expression is obtained from (16) by replacing index 1 by 2 and also $h_0 - z$ by z, $h_0 - z_0$ by z_0 and $\varkappa_1$ by $\varkappa_2 = K_0\sqrt{n_2^2 - 1} = iK_0 \sin \chi_2$.

We note that in the case of real $k_1 = n_1 k_0$ and $k_2 = n_2 k_0$, when the amplitude of lateral waves decreases only as $\frac{1}{r^2}$, at sufficiently large distances from the source of radiation practically the entire field within the stratum will be determined by the field of lateral waves. This is explained by the fact that, as will be shown hereinafter, waves of the discrete spectrum are exponentially attennuated on propagation along the stratum (with the exception of some specific instances), due to withdrawal of energy from the stratum by way of leakage through boundaries. Therefore at sufficiently large distances the amplitude of waves of the discrete spectrum will be of any degree of smallness in comparison with amplitudes of lateral waves decreasing as $\frac{1}{r^2}$.

The difference in laws of attennuation of waves of the discrete spectrum and of the lateral waves can be clearly explained in the following manner. Waves of the discrete spectrum on propagation in the stratum gradually decrease in amplitude due to leakage of energy through walls of the stratum. Lateral waves on the other hand at great distances are continuously supplied with energy from the half-spaces bounding the stratum. There may exist instances, however, when the amplitude of the lateral wave will be small in comparison with amplitudes of continuous spectrum waves at any distances. This will take place for example when reflection from stratum boundaries is complete and the amplitude of discrete spectrum waves decreases only as $\frac{1}{\sqrt{r}}$, and also if in the half-spaces bounding the stratum there occurs a sufficiently large attennuation of the waves.

To substantiate the correctness of formula (16) it is necessary to meet certain conditions. In (14) we disregarded s^2 and s^4 in comparison with $1 - n_1^2$. Taking into account that expression (12) gives the maximum values s which are still of materail significance, we have a condition at which such disregarding is possible:

$$k_0 r \left|1 - n_1^2\right| \gg 1. \tag{19}$$

Further, in the analysis of integral (9), exponents of the form $e^{b(h_0 - z)}$ were taken outside the sign of integration for the value $s = 0$. This is permissible only in that case when these exponents are slowly changing functions in comparison with the exponential $e^{-k\theta rs^2}$. This takes place if h_0 is sufficiently small in comparison with r. A more precise criterion, the derivation of which we will not consider, holds that

$$\frac{h_0}{r} \ll \left|\frac{\sqrt{1-n_1^2}}{n_1}\right|. \tag{20}$$

Coefficient B_1, which determines the amplitude of lateral wave (16), can be readily found from (13) if one knows the analytic expression of the reflection coefficient $V_1(\alpha)$. (See examples in Section 4, below.) In the arbitrary case, however, B_1 can be expressed through converging series. Hereinafter, in the supplement, it is shown that

$$B_1 = -\frac{2\rho_0 k_0^2 \sqrt{1-n_1^2}\cdot F_1}{\rho_1 \left(k_0\sqrt{1-n_1^2} + E_1\right)^2}, \tag{21}$$

wherein, in the acoustic case

$$\left.\begin{aligned} F_1 &= 1 + 2\int_{-\infty}^{z_1} \frac{\rho_0}{\rho(z)} M(z)\,dz + \cdots, \\ E_1 &= i\,M(z_1) + i\int_{-\infty}^{z_1} \frac{\rho(z)}{\rho_0} M^2(z)\,dz + \cdots, \\ M(z) &= k_0^2 \int_{-\infty}^{z} \frac{\rho_0}{\rho(z)} (n^2 - n_1^2)\,dz. \end{aligned}\right\} \qquad (22)$$

Here we have written only two terms of each F_1 and E_1. The further terms, of more complex form, can be obtained if necessary by a procedure stated in the supplement.

In the electromagnetic case, formulas (21) and (22) hold, but in lieu of $\frac{\rho(z)}{\rho_0}$ and $\frac{\rho_0}{\rho_1}$, there must be substituted $n^2(z)$ and $\frac{1}{n_1^2}$, respectively. (The sign of root $\sqrt{1 - n_1^2}$, which is part of (21), must be chosen from condition $Re\sqrt{1 - n_1^2} > 0$. This follows from the fact that the root is derived from equation $(\sin\alpha)_{s=0} = (\sqrt{1 - n_1^2 + s^2})_{s=0} = \sqrt{1 - n_1^2}$. While function $\sin\alpha$ taken at the branch point (which is the meaning of $s = 0$) has a positive real portion.)

In an analogous manner is obtained the expression for the quantity B_2 which determines the amplitude of lateral wave, propagated above the stratum, only the index 1, in (21) and (22) is replaced by 2, while integration with respect to z from $-\infty$ to z, is replaced by integration from $+\infty$ to z.

In the case of arbitrary functions $n(z)$ and $\rho(z)$ characterizing the media bounding the stratum below and above, calculation of E_1, F_2 and E_2, F_2 requires numerical integration.

So far we have considered the case $n_1 \neq 1$, $n_2 \neq 1$. Now let $n_1 = 1$ but $n_2 \neq 1$. In such a case $\sqrt{n_1^2 - \cos^2\alpha} = \sin\alpha$ and $V_1(\alpha)$, unlike $V_2(\alpha)$, will be a single value function of α. Branch point A_1 will be absent. The continuous spectrum will be given first by the integral along edges of the section extending from branch point A_2, and secondly by integral along path T_2 of Figure 1, which now will be different from zero (in the demonstration of the equality of this integral with zero, material use was made of the fact that $n_1 \neq 1$, $n_2 \neq 1$). The first of these will yield the lateral wave studied above (expression (16) with index 1 being replaced by index 2), while the second -- the lateral wave induced by propagation in the half-space bounding the stratum from below. We obtain the analytical expression of this last-named wave. As the initial expression we have:

$$W_1 = -\frac{i k_0}{2} \int_0^{+i\infty} \Phi(\alpha) \; H_0^{(1)}(k_0 r \cos\alpha) \cos\alpha \, d\alpha, \qquad (23)$$

Dividing this integral in two, from $-i\infty$ to 0 and from 0 to $+i\infty$, and replacing in the first of them α by $-\alpha$, we obtain:

$$W_1 = -\frac{i k_0}{2} \int_0^{i\infty} (\Phi + \Phi^+) \, H_0^{(1)}(k_0 r \cos\alpha) \cos\alpha \, d\alpha,$$

wherein for brevity the denotation is used $\Phi = \Phi(\alpha)$ and $\Phi^+ = \Phi(-\alpha)$.

Substituting therein expression (2) for $\Phi(\alpha)$, denoting $V_1(-\alpha) = V_1^+$, and taking into account that $V_2(-\alpha) = \frac{1}{V_2(\alpha)}$

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(see remark on page 516) we have:

$$W_1 = -\frac{ik_0}{2} \times$$

$$\times \int_0^{i\infty} \frac{(1 - V_1 V_1^+)(V_2 e^{b(h_0 - z)} + e^{-b(h_0 - z)})(V_2 e^{b(h_0 - z_0)} + e^{-b(h_0 - z_0)})}{(-e^{-bh_0} - V_1 V_2 e^{bh_0})(V_2 e^{bh_0} - V_1^+ e^{-bh_0})} \times$$

$$\times H_0^{(1)}(k_0 r \cos\alpha) \cos\alpha \, d\alpha . \qquad (24)$$

This integral is calculated analogously to integral (8), the most essential region of the integration path being the region of small $|\alpha|$. The entire expression within the sign of integration, with the exception of the Hankel function which is replaced by its asymptotic value, is expressed as a power series in α .

Let us consider the coefficients of reflections $V_1(\alpha)$ and $V_2(\alpha)$ at small values of α . Later on in the supplement it is shown that on disregarding small α^3 , $V_1(\alpha)$ can be written in the form:

a) when $n_1 \neq 1$ $\qquad V_1(\alpha) = -e^{-2p_1 \alpha}$ (25)

b) when $n_1 = 1$ $\qquad V_1(\alpha) = e^{2ic_1 \alpha - 2d_1 \alpha^2}$ (25')

wherein~~g~~ p_1, c_1, and d_1 are constants.

The quantity p_1 is as a rule complex:

$$p_1 = p_1' - i p_1'' . \qquad (26)$$

At the same time $p_1' > 0$ since $|V_1| < 1$, and $p_1'' > 0$ since the value $2p_1''\alpha$ represents the wave phase advance on reflection which is always positive. Henceforth we will use in lieu of p_1'' the value h_1 introduced in accordance with the equation:

$$p_1'' = k_0 h_1 \tag{27}$$

The quantity h_1 can be considered as being the effective thickness, equal to the thickness of a certain homogeneous stratum on passage through which, forward and backward, the wave acquires the same phase advance as on reflection from the lower boundary of the stratum. When $n_1 = 1$, quantities c_1 and d_1 are real (in 25') if absorption is absent in the media ($n(z)$ is a real function). Phase advance in that instance is characterized by value c_1, and by analogy with (27) we postulate

$$c_1 = k_0 h_1 \tag{28}$$

In those cases when there is an analytical expression for function $V_1(\alpha)$, coefficients are found by resolution of right and left portions of equation (25) into a series of α and likening the coefficients of equal powers of α. In the general case p_1, c_1, and d_1 can be written as a series analogous to (22) (see below, in Supplement).

Formulas analogous to (25) -- (28) will obtain, naturally, also for $V_2(\alpha)$.

Integral (24) relates to the instance $n_1 = 1$, $n_2 \neq 1$. Consequently we ~~[illegible]~~ use formula (25') for V_1 and the formula $V_2 = -e^{-2p_2\alpha}$, analogous to (25), for V_2. As a result we have:

$$1 - V_1 V_1^+ = 1 - e^{-4d_1\alpha^2} \simeq 4d_1\alpha^2$$

At the same time $p_1' > 0$ since $|V_1| < 1$, and $p_1'' > 0$ since the value $2p_1''\alpha$ represents the wave phase advance on reflection which is always positive. Henceforth we will use in lieu of p_1'' the value h_1 introduced in accordance with the equation:

$$p_1'' = k_0 h_1 \tag{27}$$

The quantity h_1 can be considered as being the effective thickness, equal to the thickness of a certain homogeneous stratum on passage through which, forward and backward, the wave acquires the same phase advance as on reflection from the lower boundary of the stratum. When $n_1 = 1$, quantities c_1 and d_1 are real (in 25') if absorption is absent in the media ($n(z)$ is a real function). Phase advance in that instance is characterized by value c_1, and by analogy with (27) we postulate

$$c_1 = k_0 h_1 \tag{28}$$

In those cases when there is an analytical expression for function $V_1(\alpha)$, coefficients are found by resolution of right and left portions of equation (25) into a series of α and likening the coefficients of equal powers of α. In the general case p_1, c_1, and d_1 can be written as a series analogous to (22) (see below, in Supplement).

Formulas analogous to (25) -- (28) will obtain, naturally, also for $V_2(\alpha)$.

Integral (24) relates to the instance $n_1 = 1$, $n_2 \neq 1$. Consequently we ~~cannot~~ use formula (25') for V_1 and the formula $V_2 = -e^{-2p_2\alpha}$, analogous to (25), for V_2. As a result we have:

$$1 - V_1 V_1^* = 1 - e^{-4d_1\alpha^2} \approx 4d_1\alpha^2$$

In an analogous manner are expanded in series in terms of α the other expressions within the sign of integration. Using the asymptotic value (9) of the Hankel function and taking into account that $\cos\alpha \approx 1 - \frac{\alpha^2}{2}$, we obtain an integral of the form (15), which gives as final result :

$$W_1 = \frac{2 i d_1}{k_0 r^2} \cdot \frac{\left(h_0 + h_2 - z + \frac{i p_2'}{k_0}\right)\left(h_0 + h_2 - z_0 + \frac{i p_2'}{k_0}\right)}{\left(h + \frac{i p_2'}{k_0}\right)^2} e^{i k_0 r} \qquad (29)$$

wherein h denotes the total effective thickness of the stratum,

$$h = h_0 + h_1 + h_2 \, . \qquad (30)$$

As we can see, dependency of the amplitude of lateral wave on 4 in this instance is of the same type as in the case of $n_1 \neq 1$ (see (16)); the dependency on z is, however, a different one.

In lieu of conditions (19) and (20) characterizing the usability of computation, we have here one condition, namely:

$$\frac{k_0 h}{\sqrt{k_0 r}} \ll 1 \qquad (31)$$

If not only $n_1 = 1$ but also $n_2 = 1$, then in the complex region α there will be not a single branch point. Whereas the integral along the imaginary axis yields two lateral waves, of which one is given by the expression:

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$$W_1 = \frac{2id_1}{k_0 r^2 h^2}(h_0 + h_2 - z)(h_0 + h_2 - z_0)e^{ik_0 r}, \qquad (29')$$

where

$$h = h_0 + h_1 + h_2, \quad h_1 = \frac{c_1}{k_0}, \quad h_2 = \frac{c_2}{h_0},$$

while the other is obtained from it on replacing index 1 by 2, $h_0 - z$ by z, and $h_0 - z_0$ by z_0.

It is also of interest to note the case when $n_1 = n_2 \neq 1$. Under this condition the integral along the imaginary axis vanishes, and both lateral waves are obtained from the integral along the edges of the single section, extending from branch point $\cos\alpha = n_1$. The analytic expression for one of them coincides with (16), while for the other it is obtained from (16) on performing the above-indicated replacement.

Thus, two lateral waves are always present. One of them is dependent on propagation in the lower half-space, and the other in the upper. For the wave dependent, for example, on propagation in the lower half-space, we obtain different analytical expressions (16) and (20) depending on whether or not n_1 is equal to unity. The case wherein n_1 is anywhere proximate to unity but distinct therefrom, we cannot study in view of the necessity of meeting condition (19).

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Section 3. Discrete Spectrum

As was stated in Section 1, the discrete spectrum is given by the sum of differences of the expression within the sign of integration in (1). Here we will study it in the case where thickness of the homogeneous stratum containing the source of radiation and the receiver is large in comparison with the wave length.

Location of the poles is determined by roots of equation (5). Let us assume in this equation

$$V_1(\alpha) = -e^{i\varphi_1(\alpha)}, \qquad (32)$$

wherein

$$\varphi_1(\alpha) = -i \ln(-V_1),$$

and analogously for $V_2(\alpha)$. Then equation (5) can be written in the form:

$$e^{2ik_0h_0 \sin\alpha + i(\varphi_1 + \varphi_2)} = 1$$

or

$$2k_0h_0 \sin\alpha + \varphi = 2\pi l \qquad (33)$$

wherein l is a whole number and $\varphi = \varphi_1 + \varphi_2 = i \ln V_1V_2$

Let us consider first the case where l is large. Then disregarding φ in comparison with $2\pi l$ we obtain from (33) in first approximation the following series of solutions:

$$\alpha_l^{(1)} = \arcsin \frac{\pi l}{k_0h_0}. \qquad (34)$$

On substituting this solution into the small term ϕ of (33), we have in the next approximation, which will suffice:

$$\alpha = \arcsin\left[\frac{\pi l}{k_0 h_0} - \frac{\phi(\alpha_l^{(1)})}{2k_0 h_0}\right] \simeq \arcsin\frac{\pi l}{k_0 h_0} + i\,\frac{\ln\left[V_1(\alpha_l^{(1)})\,V_2(\alpha_l^{(1)})\right]}{2\sqrt{(k_0 h_0)^2 - (\pi l)^2}} \tag{35}$$

As will be shown hereinafter, attenuation of each wave of the discrete spectrum will be determined by the imaginary portion of the corresponding root of α_l. Using the denotation

$$\Delta_l = \operatorname{Im} \alpha_l , \tag{36}$$

we have from (35)

$$\Delta_l = -\frac{\ln\left|V_1(\alpha_l^{(1)})\,V_2(\alpha_l^{(1)})\right|}{2\sqrt{(k_0 h_0)^2 - (\pi l)^2}} \tag{37}$$

it being assumed that $\pi l < k_0 h_0$.

The solution method utilized is not suitable for small values of α_l, which in accordance with (34) corresponds to small values of α_l when $k_o h_o$ is large. In this case we will use another method based on the fact that for small values of α we have explicit expressions (25) and (25') for the function $V_1(\alpha)$ and analogous expressions for $V_2(\alpha)$.

In so doing it is necessary to differentiate three instances.

I. $n_1 \neq 1$, $n_2 \neq 1$

Likening of (25) and (32) gives in this case $\phi_1 = 2i\rho_1\alpha$.

Analogously, for the upper boundary we have $\varphi_2 = 2ip_2\alpha$. Consequently $\varphi = \varphi_1 + \varphi_2 = 2i(p_1 + p_2)\alpha$. Substituting this value of φ in (33), and assuming therein $\sin\alpha \simeq \alpha$, we have

$$\alpha_l = \frac{\pi l}{k_0 h_0 + i(p_1 + p_2)}, \tag{38}$$

Here, in accordance with equations (26) and (27) and their analogues for the index 2

$$k_0 h_0 + i(p_1 + p_2) = k_0 h + i(p_1' + p_2'),$$

wherein h is the effective thickness of the stratum, determined by equation (30).

Denoting

$$p' = p_1' + p_2' \tag{39}$$

and assuming that $k_0 h \gg p'$, we obtain from (38)

$$\alpha_l = \frac{\pi l}{k_0 h} - i\Delta_l, \tag{40}$$

wherein

$$\Delta_l = \frac{\pi l}{(k_0 h)^2} p'. \tag{41}$$

II. $n_1 = 1$, $n_2 \neq 1$

In this case it is necessary to use expression (25') for $V_1(\alpha)$. Likening with (32) gives

$$\varphi(\alpha) = \varphi_1 + \varphi_2 = 2(c_1 + ip_2)\alpha + 2id_1\alpha^2.$$

Substituting this expression in (33) and again assuming therein $\sin\alpha \approx \alpha$, we obtain, on performing certain transformations, the formula (40), wherein

$$\Delta l = -\frac{2l(\pi l)^2}{(K_0 h)^2}\left(\beta_2' + \frac{\pi l}{K_0 h} d_1\right) \qquad (42)$$

Here again $h = h_0 + h_1 + h_2$, and $h_1 = \frac{c_1}{K_0}$, $h_2 = \frac{\beta_2''}{K_0}$.

III. $N_1 = n_2 = 1$

In the same manner as above we have

$$\Delta l = \frac{(\pi l)^2}{(K_0 h)^3} d , \qquad (43)$$

wherein

$$d = d_1 + d_2 .$$

Let us calculate now the field, given by deductions of the expression within the integration sign in (1), at the poles which have been found. (In so doing we assume that nowhere within the finite region of the complex plane α is there a point of pole concentration. Insofar as poles located within distant regions are concerne, the analysis performed in an analogous manner to that appearing in the Supplement of /1/ can be used to show that they are located on a line extending toward $\alpha = -i\infty$ and approaching the straight line $\alpha_1 = \frac{\pi}{2}$, and possess a point of concentration of the poles at infinity. This, however, does not preclude the use of the Cauchy theorem on substraction /6/. All distant poles contribute negligibly little to our solution.)

It is known that to do this we must substitute $\alpha = \nu_l$ everywhere in the above-referred expression except in the denominator, and replace in the denominator $f = 1 - V_1V_2e^{2b_\ell h_o}$ by $\left(\frac{df}{d\alpha}\right)_{\alpha_\ell}$. The result obtained on performing these substitutions must be multiplied by $2\pi i$ and thereafter summate for every ℓ. In so doing, on utilizing equation (5) for the poles, it is not difficult to show that

$$\left(\frac{df}{d\alpha}\right)_{\alpha_\ell} = \tag{44}$$

$$= \left(\frac{1}{V_1}\cdot\frac{dV_1}{d\alpha} + \frac{1}{V_2}\cdot\frac{dV_2}{d\alpha} - 2ik_0h_0\cos\alpha\right)_{\alpha_\ell}.$$

Here, on assuming k_oh_o sufficiently large, one can disregard the first two terms in comparison with the last term.

Further, the expression

$$\left[e^{-b_\ell z} + V_1(\alpha_\ell)e^{b_\ell z}\right]\left[e^{-b_\ell(h_o - z_o)} + V_2(\alpha_\ell)e^{-b_\ell(h_o - z_o)}\right]$$

can be rewritten on taking into account that $V_1(\alpha_\ell)V_2(\alpha_\ell)e^{2b_\ell h_o} = 1$, in a form symmetric with respect to z and z_o. (The reader should not be perplexed by the resulting lack of symmetry with respect to indexes 1 and w in the expressions obtained. In the same manner one could also obtain expressions of the form containing, in lieu of $V_1(\alpha)$, only $V_2(\alpha)$):

$$\frac{e^{-b_\ell h_o}}{V_1(\alpha_\ell)}\left[e^{-b_\ell z} + V_1(\alpha_\ell)e^{b_\ell z}\right]\left[e^{-b_\ell z_o} + V_1(\alpha_\ell)e^{b_\ell z_o}\right].$$

Finally, on replacing the function $H_0^{(1)}(k_0 r \cos\alpha)$ by its asymptotic value, we obtain for the discrete spectrum

$$\psi = \frac{e^{\frac{i\pi}{4}}}{h_0}\sqrt{\frac{\pi}{2k_0 r}}\sum_{\ell=1}^{\infty}\left[e^{-b_\ell z} + V_1(\alpha_\ell)e^{b_\ell z}\right]\times$$

$$\times\left[e^{-b_\ell z_0} + V_1(\alpha_\ell)e^{b_\ell z_0}\right]\frac{e^{ik_0 r\cos\alpha_\ell}}{V_1(\alpha_\ell)\cos^{3/2}\alpha_\ell}, \qquad (45)$$

where $b_\ell = ik_0h_0\sin\alpha_\ell$, while α_ℓ is given by expression (35) in the case of large number ℓ, and by the expressions (40) to (43) for the other ℓ. (The summation begins with $\ell = 1$, since for $\ell = 0$ we have from (40) $\alpha_\ell = 0$. However $\alpha = 0$ is not a pole of the expression within the integration sign of (1), because in it the numerator also becomes zero simultaneously with the denominator. Disclusure of indefiniteness gives a value equal to zero. Further, the poles corresponding to $\ell < 0$ are not located in the region $0 < \alpha_1 < \pi$ and $-\infty < \alpha_2 < 0$ and are not involved in distortion of the path of integration.) Thus the discrete spectrum is fully determined by means of coefficients of reflection of plane waves from stratum boundaries. For small α_ℓ these coefficients are determined by formulas (25) and (25') and their analogues for index 2; in the case of arbitrary α_ℓ they can be represented in the form of converging series /8/.

With increasing wave number ℓ, attenuation also increases, since according to formulas (40) to (43) under these conditions the imaginary portion of α_ℓ increases. This attenuation is due to leakage of energy through boundaries of the stratum. At sufficiently great distances an appreciable amplitude will be possessed only by waves corresponding to small ℓ, and hence also to small α_ℓ . Then formula (45) can be written in a simpler form

$$\psi = \frac{e^{ik_0 r + \frac{i\pi}{4}}\sqrt{\pi}}{h_0\sqrt{2k_0 r}} \sum_{\ell=1}^{\infty} \left[e^{-\frac{i\pi\ell z}{h}} + V_1\left(\frac{\pi\ell}{k_0 h}\right) e^{\frac{i\pi\ell z}{h}} \right] \times$$

$$\times \left[e^{-\frac{i\pi\ell z_0}{h}} + V_1\left(\frac{\pi\ell}{k_0 h}\right) e^{\frac{i\pi\ell z_0}{h}} \right] \frac{e^{-ik_0 r\frac{\alpha_\ell^2}{2}}}{V_1\left(\frac{\pi\ell}{k_0 h}\right)}, \qquad (46)$$

(On more precise consideration the factor h_0 preceding the sum in the denominator is replaced by h.)

Decrease of amplitude of the ℓ-th term with distance is determined in addition to the general factor $\frac{1}{\sqrt{r}}$, also by the exponent $e^{\frac{k_0 r}{2} \operatorname{Im} \alpha_\ell^2}$ or in view of (40) by the exponent $e^{-\frac{\pi\ell}{h} r \Delta\ell}$, wherein $\Delta\ell$ is given for different cases by the formulas (41), (42), and (43).

We have left for consideration the problem not dealt with in the foregoing paragraph relative to the poles which must be circuited in the process of distortion of the integration path on formation of lateral waves. Let us consider as an example the case when $n_1 > 1$. On distortion of path Γ_3 into Γ_4 (Figure 1), the contour enclosing the section can be drawn along line CA_1B (Figure 4), where line A_1B is selected forthwith as a path of most rapid descent. In so doing, only those poles are involved which are located below line A_1B, for instance P_2, so that we must add to the solution the deductions at these poles. Poles located above line A_1B, for instance P_1, remain as yet not involved.

Figure 4. Deformation of the path of integration Γ_3 in the case when $n_1 > 1$. The dotted line represents the section extending from branch point A_1

For calculation of integral along CA_1, the path CA_1 by a continuous distortion is converted to path of most rapid descent A_1B, changing at the same time the direction of integration into the opposite one, after which the integral is evaluated as is shown in Section 2. In so doing we must circuit pole P_1 and the other poles located above line A_1B. As a result there are circuited counter-clockwise, once, all the poles located in the region $0 < \alpha_1 < \frac{\pi}{2}$ and $\alpha_2 < 0$, the deductions at which give on summation the discrete spectrum which we have written in the form of formula (45). Thus all of the results stated herinbefore remain unchanged.

It ensues therefrom only the following circumstance not previously noted: deductions must be made only at those poles located above A_1B, where $\mathrm{Re}\sqrt{n_1^2-\cos^2\alpha} > 0$, and only at those located below A_1B, where $\mathrm{Re}\sqrt{n_1^2-\cos^2\alpha} < 0$. This follows from the fact that along path CA_1, $\mathrm{Re}\sqrt{n_1^2-\cos^2\alpha} > 0$. (See remark on page 518). This sign of the real portion of the root is retained also on continuous deformation of CA_1 into BA_1, even though the imaginary portion passes through zero, since we cross the section, and changes its sign. Conversely, on path A_1B, $\mathrm{Re}\sqrt{n_1^2-\cos^2\alpha} < 0$.

In practice, the poles located below A_1B apparently have no substantial significance since they possess a considerable imaginary portion and the discrete spectrum waves which correspond to them will undergo rapid attenuation with distance.

Section 4. Examples

We have seen hereinbefore that the acoustic and electromagnetic fields at each point of the stratum are determined in a simple manner by means of coefficients of reflection of plane waves from stratum boundaries. In the gneral case of arbitrary stratified media bounding the stratum from above and below, these coefficients are presented in the form of series (see/8/, and also the Supplement). In some instances they are known in the finite form /7/.

We will consider here, as examples, the two simplest cases.

I. Stratum bounded above and below by a homogeneous half-space

In such a case

$$V_1(\alpha) = \frac{m_1 \sin\alpha - \sqrt{n_1^2 - \cos^2\alpha}}{m_1 \sin\alpha + \sqrt{n_1^2 - \cos^2\alpha}} \qquad (47)$$

and analogously for $V_2(\alpha)$. Here n_1 is the index of refraction in the lower half-space. For the sake of definiteness we will assume $n_1 > 1$. Further, $m_1 \equiv \frac{\rho_1}{\rho_0}$ in acoustics and $m_1 \equiv n_1^2$ in electrodynamics.

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Resolution by powers of α has the form:

$$V_1(\alpha) = 1 - \frac{2m_1\alpha}{\sqrt{m_1^2-1}} + \frac{2m_1^2\alpha^2}{m_1^2-1} + \tag{48}$$

$$+ \frac{2m_1}{(m_1^2-1)^{3/2}}\left(\frac{1}{2} - m_1^2 + \frac{m_1^2-1}{6}\right)\alpha^3 + \cdots$$

Disregarding α^3 and comparing (48) and (25) we have:

$$p_1 = \frac{m_1}{\sqrt{m_1^2-1}} \tag{49}$$

The discrete spectrum is given by expression (45) or at sufficiently great distances by the expression (46), wherein α_ℓ for not small ℓ (say $\ell > 2$) is given by formula (35), and for small ℓ by the formulas (40) and (41). At the same time

$$p' = \mathrm{Re}\left(\frac{m_1}{\sqrt{m_1^2-1}} + \frac{m_2}{\sqrt{m_2^2-1}}\right). \tag{50}$$

At distances

$$r \geqslant \left(\frac{K_0 h}{\pi}\right)^2 \frac{h}{2p'} \tag{51}$$

of the entire discrete spectrum, an appreciable amplitude will be had only by one wave which is most slowly attenuated, and is defined by the term corresponding to $\ell = 1$, the expression of which according to (40), (41), and (46) is:

$$\frac{e^{ik_0 r - \frac{i\pi}{4}}\sqrt{\pi}}{h\sqrt{2k_0 r}\, V_1\left(\frac{\pi}{k_0 h}\right)}\left[e^{-\frac{i\pi z}{h}} + V_1\left(\frac{\pi}{k_0 h}\right)e^{\frac{i\pi z}{h}}\right] \times$$

$$\times\left[e^{-\frac{i\pi z_0}{h}} + V_1\left(\frac{\pi}{k_0 h}\right)e^{\frac{i\pi z_0}{h}}\right]e^{-\left(\frac{\pi}{k_0 h}\right)^2 \frac{\beta'}{h} r}, \qquad (52)$$

On the other hand, resolution of $V_1(\alpha)$ by powers of $\sqrt{n_1^2 - \cos^2\alpha} = x$ (with $\sin\alpha = \sqrt{1 - n_1^2 + x^2}$) gives:

$$V_1(\alpha) = 1 - \frac{2\sqrt{m_1^2 - \cos^2\alpha}}{m_1\sqrt{1 - m_1^2}},$$

wherefrom we obtain in accordance with (13) for the coefficient B_1, which determines the amplitude of the lower lateral wave, the expression:

$$B_1 = -\frac{2}{m_1\sqrt{1 - m_1^2}}, \qquad (53)$$

As a result, in accordance to (16) the lateral wave by order of magnitude will be:

$$W_1 \sim \frac{2n_1 e^{ik_0 m_1 r}}{m_1(1 - m_1^2)k_0 r^2} \qquad (54)$$

An analogous expression is obtained for the upper lateral wave W_2. At distances r, where

$$r \sim \frac{h}{\beta'}\left(\frac{k_0 h}{\pi}\right)^2 \ln\left(\sqrt{k_0 r}\,\frac{r}{h}\right) \qquad (55)$$

the lateral waves become equal in amplitude with the last wave of the discrete spectrum, and at greater distances exceed it. It is being assumed that n_1 and n_2 are real.

II. Stratum Formed Between Two Plates of Thickness, δ_1 and δ_2, Placed parallel to Each Other in a Homogeneous Medium

In electrodynamics this is a plane waveguide with walls of finite thickness. The coefficient of reflection is then /9/:

$$V_1(\alpha) = \frac{m_1^2 \sin^2\alpha - n_1^2 + \cos^2\alpha}{m_1^2 \sin^2\alpha - n_1^2 + \cos^2\alpha + 2 i m_1 \sin\alpha \sqrt{n_1^2-\cos^2\alpha}\, \operatorname{ctg}\left(k_0 \delta_1 \sqrt{n_1^2-\cos^2\alpha}\right)} \tag{56}$$

where m_1 and n_2 have the same significance as above and characterize the material of the plate.

Expansion in power series of α is of the form:

$$-V_1(\alpha) = 1 - \frac{2 i m_1}{a_1} \operatorname{ctg} k_0 \delta_1 a_1 \alpha - \frac{2 m_1^2}{a_1^2}\left(1 + 2 \operatorname{ctg}^2 k_0 \delta_1 a_1\right) \alpha^2 +$$

$$+ \left[\frac{i m_1}{a_1^3}\left(6 m_1^2 + 1\right) \operatorname{ctg} k_0 \delta_1 a_1 + \frac{8 i m_1^3}{a_1^3} \operatorname{ctg}^3 k_0 \delta_1 a_1 + \right.$$

$$\left. + \frac{i m_1 k_0 \delta_1}{a_1^2 \sin^2 k_0 \delta_1 a_1} + \frac{i m_1}{3 a_1} \operatorname{ctg} k_0 \delta_1 a_1 \right] \alpha^3 + \ldots,$$

wherein $a_1 = \sqrt{n_1^2 - 1}$.

Disregarding the α^3 term and comparing with (25') gives:

$$c_1 = -\frac{m_1}{a_1}\,\mathrm{ctg}\,k_0\delta_1 a_1, \qquad d_1 = \frac{m_1^2}{a_1^2 \sin^2 k_0\delta_1 a_1}$$

and analogously for c_2 and d_2 of the upper boundary. The discrete spectrum is again determined by formula (45) where α_l is formed from (35), (40), and (43).

The lower lateral wave is given by expression (29'). The amplitude, as expected tends toward zero if attenuation is present in the lower plate (n_1 is complex, for example, a conducting wall of a waveguide in the case of electrodynamics), while the thickness of the plate δ_1 increases without bounds. In practice the lateral wave will have a zero amplitude if the thickness of the wall is greater than that of the skin-layer.

Supplement

Let us derive formulas (25) and (25'), giving an explicit expression for the coefficient of reflection from an arbitrarily plane-stratified medium in the case of small angles of slide α. We assume the presence of a stratified inhomogeneous half-space having a lower boundary $z = -\infty$ and an upper boundary $z = z_1$, and led us consider first the case where $n_1 \neq 1$.

In accordance with equations (III) and (IV) of Supplement to /1/ (see also /8/), the reflection coefficient can be written as:

$$V_1(\alpha) = \frac{g_0 u(z_1) - g_1}{g_0 u(z_1) + g_1}, \qquad \text{(I)}$$

while function u(z) is determined from equation:

$$\frac{du}{dz} = im\frac{q_1}{q}\left(1 - \frac{q^2}{q_1^2}u^2\right) \qquad \text{(II)}$$

and the boundary condition $z \to -\infty$, $u \to 1$. Herein m = = m(z) and q = q(z) are functions of coordinate z determined by the correlations:

$$m(z) = K_0\sqrt{n^2(z) - \cos^2\alpha} \qquad \text{(III)}$$

$$q(z) = \begin{cases} \dfrac{\rho_0\, m(z)}{\rho(z)} & \text{in acoustics} \\ \dfrac{m(z)}{n^2(z)} & \text{in electrodynamics} \end{cases} \qquad \text{(IV)}$$

where n(z) and ρ (z) are variables in index of refraction and density of the medium when $z \leq z_1$, ρ_0 is the density of the homogeneous medium where $z > z_1$. Further, q_0 and q_1 are used to denote:

$$q_0 = q(z_1) = m(z_1) = K_0 \sin\alpha, \qquad \text{(V)}$$

$$q_1 = q(-\infty) = \begin{cases} K_0\dfrac{\rho_0}{\rho_1}\sqrt{n_1^2 - \cos^2\alpha} & \text{in acoustics} \\ \dfrac{K_0}{n_1^2}\sqrt{n_1^2 - \cos^2\alpha} & \text{in electrodynamics} \end{cases} \qquad \text{(VI)}$$

On expanding $V_1(\alpha)$ in power series of α, and disregarding α^3, it is sufficient, as is apparent from (I), to know exwhile function u(z) is determined from equation:

$$\frac{du}{dz} = im\frac{q_1}{q}\left(1 - \frac{q^2}{q_1^2}u^2\right) \qquad \text{(II)}$$

and the boundary condition $z \rightarrow -\infty$, $u \rightarrow 1$. Herein m = = m(z) and q = q(z) are functions of coordinate z determined by the correlations:

$$m(z) = K_0\sqrt{n^2(z) - \cos^2\alpha} \qquad \text{(III)}$$

$$q(z) = \begin{cases} \dfrac{\rho_0 m(z)}{\rho(z)} & \text{in acoustics} \\ & \text{to} \\ \dfrac{m(z)}{n^2(z)} & \text{in electrodynamics} \end{cases} \qquad \text{(IV)}$$

where n(z) and ρ (z) are variables in index of refraction and density of the medium when $z \leq z_1$, ρ_0 is the density of the homogeneous medium where $z > z_1$. Further, q_0 and q_1 are used to denote:

$$q_0 = q(z_1) = m(z_1) = K_0 \sin\alpha, \qquad \text{(V)}$$

$$q_1 = q(-\infty) = \begin{cases} K_0\dfrac{\rho_0}{\rho_1}\sqrt{n_1^2 - \cos^2\alpha} & \text{in acoustics} \\ \dfrac{K_0}{n_1^2}\sqrt{n_1^2 - \cos^2\alpha} & \text{in electrodynamics} \end{cases} \qquad \text{(VI)}$$

On expanding $V_1(\alpha)$ in power series of α, and disregarding α^3, it is sufficient, as is apparent from (I), to know expansion of $u(z_1)$ into a series with an accuracy up to α. Since from (II) to (LV) it follows that $u(z)$ expands by even powers of α, one must postulate generally that $\alpha = 0$. Having assumed in (II) $\alpha = 0$, and solving this equation by the method of successive approximations, considering at zero approximation the right portion of the equation equal to zero (for details see /8/), we have in the acoustic case:

$$u(z_1) = 1 + ik_0\sqrt{n_1^2-1}\, I(z) - k_0^2 \int_{-\infty}^{z_1} \frac{2\rho_1}{\rho}(n^2-1)\, I(z)\, dz -$$

$$- ik_0^3\sqrt{n_1^2-1} \int_{-\infty}^{z_1} \frac{\rho_1}{\rho}(n^2-1)\, I^2(z)\, dz + \dots \qquad \text{(VII)}$$

wherein

$$I(z) = \int_{-\infty}^{z} \left(\frac{\rho_1}{\rho} \cdot \frac{n^2-1}{n_1^2-1} - \frac{\rho}{\rho_1} \right) dz,$$

This series could be continued for any length. It can be shown that it converges, and the convergence is the more rapid the thinner the layers containing the inhomogeneous components of the medium.

On substituting now the quantity $u(z_1)$ into (I) and expanding in series in terms of α, we obtain within an accuracy of α^2 inclusive

$$V_1(\alpha) = -(1 - 2p_1\alpha + 2p_1^2\alpha^2) = -e^{-2p_1\alpha}, \qquad \text{(VIII)}$$

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where

$$p_1 = \frac{\rho_1 u(z_1)}{\rho_0 \sqrt{n_1^2 - 1}},$$

whereupon formula (25) can be considered as proven.

In the electromagnetic case in the course of all computations and in the terminal formula, $\frac{\rho_1}{\rho_0}$ must be replaced by n_1^2, and $\frac{\rho_1}{\rho}$ by $\frac{n_1^2}{n^2(z)}$. For the coefficient of reflection from the upper half-space $z_2 < z < +\infty$, we obtain the same thing, but with the substitution of index 2 for index for 1, and the replacement throughout of integral $\int_{-\infty}^{z}$ by the integral $\int_{+\infty}^{z}$.

In the case $n_1 = 1$, in lieu of equations (I) and (II) it is convenient to use other equations obtained from (I) and (II) by replacing $u(z) = \frac{q_1}{\eta(z)}$, where $\eta(z)$ is a new unknown function. In so doing we have

$$V_1(\alpha) = \frac{q_0 - \eta(z_1)}{q_0 + \eta(z_1)}. \qquad (IX)$$

We find $\eta(z)$ from the equation

$$\frac{d\eta}{dz} = \frac{im}{q}(q^2 - \eta^2) \qquad (X)$$

with the boundary condition $z \to -\infty$ and $\eta \to q_1$.

Integration of (X) by the method of successive approximations gives

$$\eta(z_1) = q_1 + iM(z_1) + \qquad (XI)$$

$$+ 2q_1 \int_{-\infty}^{z_1} \frac{m}{q} M(z)\,dz +$$

$$+ i \int_{-\infty}^{z_1} \frac{m}{q} M^2(z)\,dz + \ldots$$

where

$$M(z) = \int_{-\infty}^{z} \frac{m}{q} \left(q^2 - q_1^2 \right) dz \, ,$$

Since when $n_1 = 1$ we have $q_1 = k_0 \frac{\rho_0}{\rho_1} \sin\alpha$, the first and the third terms in the expression for $\eta(z_1)$ give terms $\sim \alpha$. On substituting (XI) in (IX), taking into account (V) and (VI) and expanding the result in series in terms of α , we obtain formula (25), wherein

$$c_1 = \frac{k_0}{A_1} \, , \qquad d_1 = \frac{D_1 k_0}{A_1^2} \, , \tag{XII}$$

where

$$A_1 = M_0(z_1) + \int_{-\infty}^{z_1} \frac{\rho}{\rho_0} M_0^2(z) \, dz + \cdots \, ,$$

$$D_1 = k_0 \frac{\rho_0}{\rho_1} \left(1 + 2 \int_{-\infty}^{z_1} \frac{\rho}{\rho_0} M_0 \, dz + \cdots \right) , \tag{XIII}$$

$$M_0(z) = k_0^2 \int_{-\infty}^{z} \frac{\rho_0}{\rho} \left(n^2 - 1 \right) dz \, .$$

Transition to the electromagnetic case is effected as usual by substituting for $\frac{\rho(z)}{\rho_0}$, $n^2(z)$.

There remains the derivation of formulas (21), (22) for the coefficient B_1 which characterizes the amplitude of the lateral wave. According to (13) it is determined from expanding $V_1(\alpha)$ and

and $V_1^+(\alpha)$ in power series in terms of $\sqrt{n_1^2 - \cos^2\alpha} = \frac{k_0 \rho_1}{\rho_0} q_1$.

To do this it is convenient to utilize formula (IX) where $\eta(z_1)$ is taken from (XI). As a result, there are obtained, the same as above, the formulas (21) and (22).

Supplement Added on Proofreading

The assumption set forth in footnote to page 525 relative to the absence of a point of pole concentration in the finite region of plane α, had appeared to us as not being questionable. However, since this question was involved in the discussion following presentation of our paper at the Session of the Department of Physico-Mathematical Sciences held on 26 September 1949 (statement of N. V. Zvolinskiy) we deem it appropriate to point out in the present supplement that the absence of such a point can be substantiated if the following limitations (in the acoustic case) are made for functions $\rho(z)$ and $n(z)$:

(1) Both functions are finite and continuous for all values of z.

(2) Function $\rho(z)$ is never equal to zero.

(3) For all values of z the functions have a first derivative with respect to z.

(4) The functions tend toward constant values when z

These conditions are fulfilled in all physically attainable cases. (Instances where functions $\rho(z)$ and $n(z)$ display sudden changes signify in all physical problems that these sudden changes are corrective transitions from rapidly-changing, smoothly-varying functions.)

To prove the absence of a point of concentration of the poles under these conditions it is sufficient to show that the left hand portion of pole equation (5) does not have a point of pole concentration in the finite region α. This in turn will be proved if we can show that neither $V_1(\alpha)$ nor $V_2(\alpha)$ have in that region any essentially-special point. This can be done on using for example for $V_1(\alpha)$ the expression (I) of the Supplement, and replacing equation (II) of function u(z) by a system of two linear equations of the first order, which are then solved by converging series utilizing the method of successive approximations of Picard.

We take the liberty of writing the final result for $V_1(\alpha)$:

$$V_1(\alpha) = \frac{k_0 \sin\alpha\, w - v}{k_0 \sin\alpha\, w + v},$$

wherein

$$w = 1 + g_1\int_{z'}^{z_1}\zeta\,dz + \int_{z'}^{z_1}\zeta\,dz\int_{z'}^{z}\eta\,dz + g_1\int_{z'}^{z_1}\zeta\,dz\int_{z'}^{z}\eta\,dz\int_{z'}^{z}\zeta\,dz + \cdots \quad (A)$$

$$v = g_1 + \int_{z'}^{z_1}\eta\,dz + g_1\int_{z'}^{z_1}\eta\,dz\int_{z'}^{z}\zeta\,dz + \int_{z'}^{z_1}\eta\,dz\int_{z'}^{z}\zeta\,dz\int_{z'}^{z}\eta\,dz + \cdots \quad (B)$$

$$\eta \equiv \eta(z) \equiv \frac{k_0^2\rho_0}{\rho(z)}\left[n^2(z) - \cos^2\alpha\right], \quad \zeta \equiv \zeta(z) \equiv \frac{i\rho(z)}{\rho_0};$$

z' denotes a finite, but sufficiently remote from $z = z_1$, coordinate, whereat the medium can already be considered as being homogeneous.

Since the series set forth converges for finite α, it follows that functions w and v are limited throughout the finite region and have as special point only branch point $\alpha = \arc\cos n_1$. It follows therefrom that $V_1(\alpha)$ has no essentially special point.

Series of the form (A) and (B) were previously obtained by B. D. Tartakovskiy.

In conclusion I wish to express on this occasion my gratitude to V. A. Fok for a number of valuable suggestions.

Literature Cited

1. Brekhovskikh, L. M., Izvestiya Akademii Nauk, SSSR, Seriya Fizicheskaye 13, No 5, 505, 1949.

2. Krasnushkin, P. Ye., Metod normal'nykh voln v primenenii k probleme dal'nykh radiosvyazey /Normal waves method in its application to distant radio communications/ MGU, 1947; ZhTF/Zhurnal Teoreticheskey Fiziki/, 18, 431, 1948.

3. Brekhovskikh, L. M., Izvestiya Akademii Nauk, SSSR, Seriya Fizicheskaya, 10, No 5-6, 491, 1946.

4. Brekhovskikh, L. M., ZhTF, 18, 455, 1948.

5. Smirnov, V. I., Advanced Mathematics Vol. III, ONTU, 1934.

6. Kurant-Gurvits, Geometricheskaya teoriya funktsiy kompleksnago peremennago, ONTU, 1934. See footnote of N. Ye. Kochin on page 85.

7. Al'pert, Ya. L. Rasprostraneniye radiovoln v ionosfere (Propagation of radiowaves in the ionosphere), GTTI, 1947.

8. Brekhovskikh, L. M., ZhTF, 19, No 10, 1126, 1949.

9. Rayleigh, Theory of Sound, II, Section 271, GTTI, 1944.

III. AVERAGE LAWS OF ATTENUATION

In the second part of the present paper /1/ we have studied in detail the field of a radiator of electromagnetic or sound waves in a stratum bounded by two arbitrary stratified inhomogeneous media. We have shown that the field at any point of the stratum is composed of a series of waves the amplitudes of which decrease with distance in accordance with the law $\frac{e^{-\beta_\ell r}}{\sqrt{r}}$, $\ell = 1,2,\ldots$ (the so-called discrete spectrum) and of two lateral waves the amplitudes of which decrease according to the law $\frac{e^{-\gamma_i r}}{r^2}$, $i = 1,2$. (Here β_ℓ is used to denote $\operatorname{Im} \cos \alpha_\ell$, and γ_i denotes $k_o \operatorname{Im} n_1$ /see formulas (45) and (16) in /1/). The damping coefficients, β_ℓ of the discrete spectrum waves and γ_i of the lateral waves are of essentially different nature. The former are due to removal of energy from the stratum by way of leakage through its boundaries (it being assumed that absorption within the medium constituting the stratum is low), while the latter are due to absorption of waves in the media bounding the stratum.

At a sufficiently great distance from the source of radiation the field is determined by the least-damped wave of the discrete spectrum or by lateral waves, and can be rated by means of formulas shown in /1/. At smaller distances from the source of radiation other waves of the discrete spectrum also become important, the number of such waves being very large if the thickness of the stratum is great, namely of the order of twice the ratio of stratum thickness to wave length. At such distances the field is practically fully determined by such a set of a large number of waVEs, while the lateral wave contributes but a negligibly small component thereto. (With an impulse regimen of the source of radiation it can still be registered due to the time difference of its ingress in comparison with those of other waves. We do not, however, consider this instance. In many cases of practical importance, the interval of distances at which the field is determined by such a set of waves is of essential significance. At the same time the field within the stratum possesses a complex interference structure, caused by superposition of all these waves. Although calculation of amplitude and phase of each wave at an arbitrary point of the stratum on the basis of /1/ presents no difficulties, the summation of all of them and analysis of the dependence of this sum on the coordinates of point is most intricate.

In such a case, of greatest interest is the determination of a certain average dependency of acoustic pressure or intensity of the electromagnetic field on distance. On averaging the complex picutre of interference maxima and minima becomes blurred. This "fine structure of the field" can all the more be omitted from theoretical consideration since it cannot be observed experimentally in most of the instances. This is due either to the unstable conditions of wave propagation as a result of fluctuations of properties of the medium, which makes it necessary to repeat the measurements and rely only on an average result, or is caused by nonmonochromatic nature of the source of radiation.

Derivation of the average laws will be the topic of the present, the third part of the instant paper.

Section 1. Average Laws of Attenuation

Let us derive the average laws of attenuation for distances sufficiently remote from the source of radiation by comparison with the thickness of the stratum (exact criterion see below, Section 3). Under such conditions the discrete spectrum is given by formula (46) (see /1/). Consider the acoustic case first. The mean square, per period, of acoustic pressure at an arbitrary point is given by the expression

$$\frac{1}{2}\rho_0^2\omega^2|\psi(r,z)|^2,$$

where ψ (r, z) is the acoustic potential, ρ_0 is the density of the medium and ω the angular frequency. This expression displays a complex dependence on coordinates characterizing the interference structure of the acoustic field. It is substantially simplified also on effecting averaging with respect to the position of the receiver, i.e., with respect to z, within the thickness of the stratum. For the thus-derived average square of acoustic pressure (we will denote it by $\overline{p^2}$) we have according to (46) (see /1/):

$$\overline{p^2} = \frac{1}{2}\rho^2\omega^2\frac{1}{h_0}\int_0^{h_0}|\psi|^2 dz = \qquad (1)$$

$$
= \frac{\pi \rho_0^2 \omega^2}{4 k_0 r h^2} \left\{ \sum_{l=1}^{\infty} \left| e^{-\frac{il\pi z_0}{h}} + V_1\left(\frac{l\pi}{k_0 h}\right) e^{\frac{il\pi z_0}{h}} \right|^2 e^{k_0 r \operatorname{Im} \alpha_l^2} I_{ll} + \right.
$$

$$
+ \sum_{l \neq m} \left[e^{-\frac{il\pi z_0}{h}} + V_1\left(\frac{l\pi}{k_0 h}\right) e^{\frac{il\pi z_0}{h}} \right] \times
$$

$$
\left. \times \left[e^{\frac{im\pi z_0}{h}} + V_1^*\left(\frac{m\pi}{k_0 h}\right) e^{-\frac{im\pi z_0}{h}} \right] e^{-\frac{ik_0 r}{2}(\alpha_l^2 - \alpha_m^2)} I_{lm} \right\},
\tag{1}
$$

wherein

$$
I_{ll} = \frac{1}{h_0} \int_0^{h_0} \left[e^{-\frac{il\pi z}{h}} + V_1\left(\frac{l\pi}{k_0 h}\right) e^{\frac{il\pi z_0}{h}} \right] \times
$$

$$
\times \left[e^{\frac{il\pi z}{h}} + V_1^*\left(\frac{l\pi}{k_0 h}\right) e^{-\frac{il\pi z}{h}} \right] dz,
\tag{2}
$$

while I_{lm} is obtained from I_{ll} , if in the second bracket of (2) within the sign of integration, the index l is replaced by m.

Let us determine this integral. From (2) we have

$$
I_{ll} = \frac{1}{h_0} \left\{ h_0 + h_0 \left| V_1\left(\frac{l\pi}{k_0 h}\right) \right|^2 + V_1\left(\frac{l\pi}{k_0 h}\right) \int_0^{h_0} e^{\frac{2il\pi z}{h}} dz + \right.
\tag{3}
$$

$$
\left. + V_1^*\left(\frac{l\pi}{k_0 h}\right) \int_0^{h_0} e^{-\frac{2il\pi z}{h}} dz \right\},
$$

We will assume (in Section 3 it will be shown that this holds for sufficiently large values of r, see below (48) and (49)), that in the sum of (1) only those terms are important which contain sufficiently small ℓ values, so that

$$\left| V_1 \left(\frac{\ell \pi}{K_0 h} \right) \right|^2 \approx 1 \quad \text{(footnote)} \tag{4}$$

(Footnote: Differentiation of the modulus of the coefficient of reflection from unity must be effected in our approximation only for the terms producing attenuation of the field with distance, and this will be done hereinafter.)

The coefficient of reflection $V_1(\alpha)$ from the lower boundary of the stratum is given in the case of sufficiently small angles α by formulas (25) and (25') (See /1/). Of these, formula (25) relates to the case when $n_1 \neq 1$, i.e., when the velocity of propagation of the waves in the stratum does not coincide with the velocity of propagation of the waves within the lower medium at a sufficient distance from the stratum, where this medium may be considered as being homogeneous. Formula (25'), on the other hand, relates to the case when these velocities coincide, i.e., when $n_1 = 1$. Condition (4), on taking into account formulas (25) and (25') can be written

$$\left. \begin{array}{ll} n \neq 1, & \beta_1' \dfrac{\ell \pi}{K_0 h} \ll 1 \\ n_1 = 1, & d_1 \left(\dfrac{\ell \pi}{K_0 h} \right)^2 \ll 1 \end{array} \right\} \tag{5}$$

The integral terms in (3) can be disregarded under certain conditions. Indeed

$$\int_0^{h_0} e^{2il\pi\frac{z}{h}}\,dz = \frac{h}{2il\pi}\left(e^{2il\pi\frac{h_0}{h}} - 1\right) =$$

$$= \frac{h}{2il\pi}\left(e^{-2il\pi\frac{h_1+h_2}{h}} - 1\right). \qquad (6)$$

wherein it is taken into account that according to (30) (see/1/)

$$h = h_0 + h_1 + h_2 . \qquad (7)$$

Expression (6) will be small in comparison with h_0, i.e., in compariston with the first term of (3), if

$$\frac{h_1 + h_2}{h} \ll 1. \qquad (8)$$

Indeed, when $l \gg 1$ it is only of the order of $\frac{h}{2\pi l}$, and when $l \sim 1$ its order of magnitude will be

$$h\left(e^{-2il\pi\frac{h_1+h_2}{h}} - 1\right) \sim 2il\pi(h_1 + h_2).$$

In both instances there is obtained a quantity that is small in comparison with h_0, and which we will disregard.

Thus, on taking into account (4), we have

$$I_{ll} = 2 \qquad (9)$$

In exactly the same manner it is shown that

$$I_{lm} = 0. \qquad (10)$$

On taking into account (4) as well as formulas (25) to (28) in /1/, we have, since $\alpha_l \approx \frac{l\pi}{K_0 h}$,

$$V_1\left(\frac{l\pi}{K_0 h}\right) \approx -e^{-2iK_0 h_1}, \tag{11}$$

In addition, according to (40) in /1/, we have

$$\alpha_l = \frac{l\pi}{K_0 h} - i\Delta_l , \tag{12}$$

where Δ_l for the different cases is given by formulas (41) to (43) in /1/.

Taking into account (9) to (12) we obtain from (1):

$$\overline{p^2} = \frac{2\pi\rho_0^2\omega^2}{K_0 r h^2}\sum_{l=1}^{\infty}\sin^2\frac{l\pi(h_1+z_0)}{h}\,e^{-2l\pi\Delta_l\frac{r}{h}} . \tag{13}$$

If we effect herein the averaging also with respect to the position of the receiver, i.e., with respect to z_0 within the limits 0, h_0, we then have:

$$\overline{p^2} = \frac{\pi\rho_0^2\omega^2}{K_0 r h^2}\sum_{l=1}^{\infty} e^{-2l\pi\Delta_l\frac{r}{h}} , \tag{14}$$

The calculations given above, in view of (5), are correct only for sufficiently small values of l . Nevertheless, the summation in (14) we extend up to $l = \infty$. This is correct if the distance r is sufficiently great, so that the terms corresponding to large ℓ values, in (14) and (13), will be negligibly small.

The quantity $\Delta \ell$ in (13) and (14) characterizes the damping of each of the waves of the discrete spectrum, with distance, which takes place because of leakage of energy through the boundaries of the stratum. We must differentiate three cases: $n_1 \neq 1, n_2 \neq 1$; $n_1 = 1, n_2 \neq 1$; $n_1 = n_2 = 1$; for which $\Delta \ell$ is given, respectively, by the formulas (41) to (43) in /1/. On substituting these formulas in (14) we obtain the following expression for the square of acoustic pressure as a function of distance in all of the three cases:

$$\bar{p}^2 = \frac{\pi \rho_0^2 \omega^2}{K_0 r h^2} \sigma(r), \tag{15}$$

where σ (r) has a different form in each of the different cases, namely:

I. $n_1 \neq 1, \; n_2 \neq 1,$

$$\sigma(r) = \sigma_1(r) = \sum_{\ell=1}^{\infty} e^{-2p'\left(\frac{\ell\pi}{K_0 h}\right)^2 \frac{r}{h}}; \tag{16}$$

II. $n_1 = 1, \; n_2 \neq 1,$

$$\sigma(r) = \sigma_2(r) = \sum_{\ell=1}^{\infty} e^{-2\left(\frac{\ell\pi}{K_0 h}\right)^2 \frac{r}{h}\left(p_2' + \frac{\ell\pi}{K_0 h} d_1\right)},$$

or, on assuming that

$$\frac{\ell\pi}{K_0 h} d_1 \ll p_2', \tag{17}$$

we have

$$\sigma_2(r) = \sum_{l=1}^{\infty} e^{-2p_2'\left(\frac{l\pi}{k_0 h}\right)^2 \frac{r}{h}}; \qquad (18)$$

III. $n_1 = n_2 = 1$,

$$\sigma(r) = \sigma_3(r) = \sum_{l=1}^{\infty} e^{-2d\left(\frac{l\pi}{k_0 h}\right)^3 \frac{r}{h}}. \qquad (19)$$

On comparing formulas (18) and (16) we see that the law of attenuation of acoustic pressure with distance is the same for the first and the second case; only in (16) the exponent contains p_2'. Since $p' = p_1' + p_2' > p_2'$ it follows that in the case II the acoustic pressure undergoes attenuation with distance more slowly than in case I. Subsequently we will see that attenuation in case III is found to take place still more slowly.

This is due to the fact that with sufficiently small angles of slide α the stratum boundary is more reflective when $n_1 = 1$, and less reflective when $n_1 \neq 1$. Indeed, according to (25) and (25') of /1/, we have $|V_1|^2 = e^{-4p_1'\alpha}$ for $n_1 \neq 1$, and $|V_1|^2 = e^{-4d_1\alpha^2}$ for $n_1 = 1$. With a sufficiently small α first expression will differ more from unity than the second. (When $n_1 = n_2 = 1$ we limit our considerations to the case when there is no absorption in the media bounding the stratum (n(z) is real). Then, as this can be seen from formulas (XI) and (XII) of Supplement to /1/, the quantities c_1 and d_1, and their analogues for the upper boundary, are real. No difficulty is involved in considering also the case of complex c_1 and d_1. Then, with a sufficiently large imaginary portion of c_1 (or of c_2 for the upper medium) the cases II and III will not differ by their attenuation law from case I.) All these considerations must relate, of course, to reflection from the upper boundary of the stratum. As a result, Case III corresponds to the most reflective boundaries, and naturally therefore the law of attenuation is found to be the slowest. In case II the lower boundary reflects better, the upper less well. Absorption in the lower boundary can in this instance be disregarded entirely in comparison with the absorption in the upper boundary, which is expressed mathematically by the condition (17). Hence in the law of attenuation (18) for this case, only the characteristic of the upper medium p_2' is present. In case I, both boundaries absorb relatively strongly, and the law of attenuation is found to be the most rapid.

Let us analyze more in detail expressions (16), (18), (19). In so doing, it is convenient to introduce non-dimensional distances, defining them as follows:

$$\left.\begin{aligned} &\text{I.} \qquad \rho_1 = 2p'\left(\frac{\pi}{k_0 h}\right)^2 \frac{r}{h}, \\ &\text{II.} \qquad \rho_2 = 2p_2'\left(\frac{\pi}{k_0 h}\right)^2 \frac{r}{h}, \\ &\text{III.} \qquad \rho_3 = 2d\left(\frac{\pi}{k_0 h}\right)^3 \frac{r}{h}. \end{aligned}\right\} \quad (20)$$

we have then,

$$\sigma_1 = \sum_1^\infty e^{-\rho_1 l^3}, \quad \sigma_2 = \sum_1^\infty e^{-\rho_2 l^2}$$

$$\sigma_3 = \sum_1^\infty e^{-\rho_3 l^3}. \tag{21}$$

Consider to begin with the first of these sums. For a sufficiently small ρ_1, it can be replaced by an integral in l from 0 to ∞, as a result of which we have

$$\sigma_1 \approx \frac{1}{2}\sqrt{\frac{\pi}{\rho_1}}. \tag{22}$$

Using the formula of Euler it can be shown that the error resulting therefrom is of the order of 1. Hence we must have

$$\sqrt{\rho_1} \ll 1. \tag{23}$$

On substituting (22) in (15) and taking into account (20) we have

$$\bar{p}^2 = \frac{\rho_0^2 \omega^2 \sqrt{\pi}}{2\sqrt{2 g' h}} \cdot \frac{1}{r^{3/2}}. \tag{24}$$

Thus at distances which satisfy condition (23), the square of acoustic pressure decreases in accordance with the law $\frac{1}{r^{3/2}}$.

At greater distances, satisfying the condition

$$\rho_1 \gg 1, \tag{25}$$

of the entire sum (16) only the first term will be of importance, and according to (15) at such distances we will have the following law of attenuation

$$\bar{p}^2 = \frac{\rho_0^2 \omega^2 \pi}{K_0 r h^2} e^{-\rho_1}. \tag{26}$$

The Figure shows a graph of function $\sigma_1(\rho)$, determined by the first expression of (21) obtained on numerical summation. Formula (22), adapted for use at distances which satisfy the condition (23), corresponds on this graph to the initial rectilinear sector of the curve. The end of the curve in the graph corresponds to the law of attenuation (26).

Drawing on page 539 of text: Graphs of functions determining the law of attenuation of acoustic pressure and intensity of electromagnetic field with distance in various cases. Along the axis of the abscissa is plotted the non-dimensional distance ρ.

The law of attenuation for case II will be the same as in case I, somewhat changed is only the definition of the non-dimensional distance.

In case III the law of attenuation is again given by formula (15) wherein now it must be assumed $\sigma = \sigma_3(\rho_3)$. While $\sigma_3(\rho_3)$ is given by the last of the expressions (21) and is also shown in the drawing. At relatively short distances, satisfying the condition

$$\sqrt[3]{\rho^3} \ll 1, \tag{27}$$

replacement of the sum by an integral gives

$$\sigma_3 = \frac{0.893}{\sqrt[3]{\rho^3}}, \tag{28}$$

On substituting σ_3 ~~om place of~~ in place of σ in (15) and taking into account (20) we obtain the law of attenuation

$$\overline{p^2} = \frac{0.893\rho_0^2\omega^2}{\sqrt[3]{2\alpha h^2}} \cdot \frac{1}{r^{4/3}}, \tag{29}$$

At greater distances $(\rho_3 \gg 1)$ the law of attenuation coincides with (26), wherein ρ_1 is replaced by ρ_3.

We will now pass to the average laws of attenuation of the electromagnetic field. The components of this field are expressed by means of the vertical component of the Hertz vector $\psi(r,z)$ in the following manner:

$$E_r = \frac{\partial^2\psi}{\partial r\,\partial z}, \quad E_z = -\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right), \quad E_\phi = 0$$

$$H_r = H_z = 0, \quad H_\phi = -ik_0\frac{\partial\psi}{\partial r}, \tag{30}$$

On substituting therein the value of ψ from formula (46) of /1/, and taking into account that $\left(\frac{l\pi}{k_0 h}\right)^2 \ll 1$ and $k_0 r \gg 1$, we have:

$$E_r = \frac{k_0\pi^{3/2}e^{ik_0 r + \frac{i\pi}{4}}}{h^2\sqrt{2k_0 r}} \sum_{l=1}^{\infty} l\left[e^{-\frac{il\pi z}{h}} - V_1\left(\frac{l\pi}{k_0 h}\right)e^{\frac{il\pi z}{h}}\right] \times \tag{31}$$

$$\times\left[e^{-\frac{il\pi z_0}{h}} + V_1\left(\frac{l\pi}{k_0 h}\right)e^{\frac{il\pi z_0}{h}}\right]\frac{e^{-ik_0 r\alpha_l^2}}{V_1\left(\frac{l\pi}{k_0 h}\right)},$$

(cont'd page 74)

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$$H_\phi \approx E_z = \frac{k_0^2\sqrt{\pi}\, e^{ik_0 r + \frac{i\pi}{4}}}{h\sqrt{2k_0 r}} \sum_{l=1}^{\infty} \left[e^{-\frac{il\pi z}{h}} + V_1\left(\frac{l\pi}{k_0 h}\right) e^{\frac{il\pi z}{h}} \right] \times$$

$$\times \left[e^{-\frac{il\pi z_0}{h}} + V_1\left(\frac{l\pi}{k_0 h}\right) e^{\frac{il\pi z_0}{h}} \right] \frac{e^{-\frac{ik_0 r}{2}\alpha^2 l}}{V_1\left(\frac{l\pi}{k_0 h}\right)},$$

In exactly the same manner as above, effective averaging with respect to thickness of the stratum, in case I ($n_1 \neq 1$, $n_2 \neq 1$), we have the following laws of attenuation:

$$\overline{E_r^2} = \frac{k_0\pi^3}{rh^4}\,\sigma_4(\rho_1) \tag{32}$$

and

$$\overline{E_z^2} = \overline{H_\phi^2} = \frac{k_0^3\pi}{rh^2}\,\sigma_1(\rho_1), \tag{33}$$

wherein

$$\sigma_4(\rho_1) = \sum_{l=1}^{\infty} l^2 e^{-\rho_1 l^3}$$

while $\sigma_1(\rho_1)$ and ρ_1 are determined in accordance with equations (20) and (21). Function $\sigma_4(\rho)$ is shown graphically in the drawing. At small values of ρ_1 satisfying the condition $\rho_1^{3/2} \ll 1$, replacement of the sum in (32) by an integral, gives:

$$\sigma_4(\rho_1) = \frac{\sqrt{\pi}}{4\rho_1^{3/2}}, \tag{34}$$

This determines, according to (32), for $\overline{E_r^2}$ a law of attenuation of the form $\frac{1}{r^{5/2}}$. For $\overline{E_z^2}$ and $\overline{H_\varphi^2}$ according to (22), the law will be $\frac{1}{r^{3/2}}$. With $\rho_1 \gg 1$ we will have

$$\sigma_4(\rho_1) \approx \sigma_7(\rho_1) \approx e^{-\rho_1}; \tag{35}$$

In case II ($n_1 \neq 1$, $n_2 \neq 1$) the laws of attenuation will be the same, only in lieu of ρ_1 must be substituted the numerical distance ρ_2.

In case III ($n_1 = n_2 = 1$) we have:

$$\overline{E_r^2} = \frac{k_0 \pi^3}{r h^4}\sigma_5(\rho_3) \tag{36}$$

and

$$\overline{E_z^2} = \overline{H_\varphi^2} = \frac{k_0^3 \pi}{r h^2}\sigma_3(\rho_3), \tag{37}$$

where

$$\sigma_5(\rho_3) = \sum_{l=1}^{\infty} l^2 e^{-\rho_3 l^3},$$

~~(38)~~

while the function $\sigma_3(\rho_3)$ is determined by expression (21) and has been studied above. With $\sqrt{\rho_3} \ll 1$ it is given by expression (28) as a result of which for $\overline{E_z^2}$ and $\overline{H_\varphi^2}$ we have the law of attenuation $\frac{1}{r^{4/3}}$.

Function $\sigma_5(\rho_3)$ has not been encountered previously. It is shown graphically in the drawing. With $\rho_3 \ll 1$ it is determined by the expression

$$\sigma_5(\rho_3) = \frac{1}{3\rho_3}, \tag{38}$$

which gives for $\overline{E_r^2}$ the law of attenuation $\frac{1}{r^2}$, which coincides with the law of attenuation in free space. When $\rho_3 \gg 1$ we have

$$\sigma_5(\rho_3) \approx \sigma_3(\rho_2) \approx e^{-\rho_3}.$$

All of the above-presented qualitative considerations pertaining to the acoustic field are automatically transferable also to the electromagnetic instance.

Section 2. Interpretation of Results from the Standpoint of Geometrical Optics

The above-derived laws of attenuation $\frac{1}{r^{3/2}}$, $\frac{1}{r^{4/3}}$, for the square of acoustic pressure and the analogous laws of the electromagnetic field ensue also from considerations based upon geometrical acoustics (optics). In the specific instance when the stratum is bounded by two homogeneous half-spaces, this problem was studied by us at an earlier date /2/. Herein we will consider the general case.

In approximation of geometrical optics, one must visualize that the point of reception is reached by the direct ray and an infinitely large number of rays, reflected a different number of times from stratum boundaries. Each of these rays can be visualized as being emitted from a certain "imaginary" source of radiation.

All the imaginary sources of radiation are disposed along a straight line, perpendicular to the boundaries of the stratum and passing through the source of radiation.

The averaging, resulting in a blurring of the interferential structure of the field, corresponds, on such an approach, to the non-coherent (energy) summation of the fields of imaginary sources. We will be concerned with the field at great distances, relative to thickness of the stratum, from the source of radiation. Under such conditions the location of source and receiver with respect to boundaries of the stratum is found to be of no importance. Therefore let us suppose for the sake of simplicity that source of radiation and receiver are in the middle of the stratum. At the same time all the imaginary sources will be located at the same distance h from one another, h being the thickness of the stratum which we assume to be equal to the effective thickness referred to in the foregoing paragraph. The distance from the point of reception to each of the imaginary sources of radiation will be equal to $R_n \sqrt{r^2 + (nh)^2}$, wherein n = 1, 2 ... Each of the sources of radiation contributes to the square of acoustic pressure at the point of reception, the following component

$$\frac{\rho_0^2 \omega^2}{2} \cdot \frac{M_1^p(\alpha_n) M_2^q(\alpha_n)}{R_n^2},$$

wherein p and q are the number of reflections of the corresponding ray from lower and upper boundary of the stratum, M_1 (α_n) and

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$M_2(\alpha_n)$ being the coefficients of reflection from these boundaries, in energy, that is

$$M_1(\alpha_n) = \left|V_1(\alpha_n)\right|^2, \quad M_2(\alpha_n) = \left|V_2(\alpha_n)\right|^2 \tag{39}$$

and finally

$$\alpha_n = \operatorname{arc\,tg} \frac{nh}{r} \tag{40}$$

is the angle of slide of the ray relative to the stratum boundaries.

(The constant coefficient depending upon the power of the source of radiation is selected here, the same as in the foregoing paragraph, in such a manner that in free space we have for the acoustic potential $\psi = \frac{1}{R} e^{i(k_0 R - \omega t)}$. Hence for the mean square of acoustic pressure we will have $\overline{p^2} = \frac{\rho_0^2 \omega^2}{2} |\psi|^2 = \frac{\rho_0^2 \omega^2}{2R^2}$.)

Effecting the summation for all imaginary sources of radiation, it is not difficult to obtain:

$$\overline{p^2} = \frac{\rho_0^2 \omega^2}{2} \sum_{n=0}^{\infty} \left[\frac{M_1^n(\alpha_{2n}) M_2^n(\alpha_{2n})}{R_{2n}^2} + \right.$$

$$+ \frac{M_1(\alpha_{2n+1}) + M_2(\alpha_{2n+1})}{R_{2n+1}^2} \times M_1^n(\alpha_{2n+1}) M_2^n(\alpha_{2n+1}) + \tag{41}$$

$$\left. + \frac{M_1^{n+1}(\alpha_{2n+2}) M_2^{n+1}(\alpha_{2n+2})}{R_{2n+2}^2} \right]$$

We will obtain analogous series for all components of the electromagnetic field, with only that difference that within the sum sign will be included angle factors characterizing the direction of the source of radiation.

At distances from the source of radiation, which are large in comparison with the thickness of the stratum, terms of series (41) will decrease very slowly. As a result, consecutive terms of the series will differ but little from one another This makes it possible for us to replace the entire expression in brackets in (41) by

$$\frac{4}{R_{2m}^2} M_1^m(\alpha_{2m}) M_2^m(\alpha_{2m})$$

and to replace the sum by an integral. As a result, we have

$$\overline{p^2} = 4\int_0^\infty \frac{M_1^m(\alpha_{2m}) M_2^m(\alpha_{2m})}{R_{2m}^2}\, dm, \tag{42}$$

With small angles of slide the coefficients of reflection M_1 and M_2 are given by formulas (39) (See above) and (25) and (25') from /1/.

Let us consider first case I.

<u>Case I. $n_1 \neq 1$, $n_2 \neq 1$</u>

Here $M_1(\alpha) = e^{-4p_1'\alpha}$ and M_2 is analogous, hence

$$M_1(\alpha) M_2(\alpha) = e^{-4p'\alpha}, \tag{43}$$

where

$$p' = p_1' + p_2'$$

On taking further into account that $\alpha_{1n} \approx \frac{2nh}{r}$, we have from (42)

$$\overline{p^2} = 2\rho_0^2\omega^2 \int_0^\infty \frac{e^{-8n^2p'\frac{h}{r}}}{r^2+4n^2h^2}\, dn \approx \tag{44}$$

$$\approx \frac{\rho_0^2\omega^2\sqrt{\pi}}{2\sqrt{2p'h}} \cdot \frac{1}{r^{3/2}},$$

which coincides with the above-obtained expression (24). In so doing we have disregarded the second term in the denominator within the sign of integration, since we assume throughout that $\alpha^2_{2n} \approx \frac{4n^2h^2}{r^2} \ll 1$.

Case II, as we have already seen, does not constitute anything qualitatively new. Let us consider Case III.

<u>Case III. $n_1 = n_2 = 1$</u>

Here, according to (39), (see above) and (25') of /1/:

$$M_1(\alpha) = e^{-4d_1\alpha^2}$$

and analogously for $M_2(\alpha)$.

Consequently

$$M_1(\alpha)\,M_2(\alpha) = e^{-4d\alpha^2},$$

and

$$d = d_1 + d_2.$$

Now we have from (42)

$$\overline{p^2} = 2\rho_0^2\omega^2 \int_0^\infty \frac{e^{-16\alpha n^2 \frac{h^2}{r^2}}}{r^2+4n^2h^2}\, dn \approx \frac{0.843\rho_0^2\omega^2}{\sqrt[3]{2\alpha h^2}} \times \frac{1}{r^{4/3}} \qquad (45)$$

which coincides with (29)

In the same manner one could obtain the average laws of attenuation, derived in the foregoing paragraph, for the electromagnetic field.

We have made the assumption that in the sum (41) and the integrals obtained therefrom, only the terms corresponding to small values of $\alpha_n \approx \frac{nh}{r}$ are important. This corresponds to a situation wherein in the entire endless chain of imaginary sources of radiation it is assumed as emitting radiations only the part nearest the basic source of radiation, the dimensions of this radiation emitting part of the chain being small in comparison with the distance to the receiver. This assumption can be substantiated in the following manner. In integral (44), due to the decrease of the exponent, the primary part is played by the quantities n which do not exceed the order of magnitude

$$n_{max} \sim \frac{1}{2}\sqrt{\frac{r}{p'h}},$$

This gives the following length for the radiating part of the chain:

(Footnote: A simple, explicit presentation can be made for the law of "three half" ($\overline{p^2} \sim \frac{1}{r^{3/2}}$). Since the length of the radiating part of the chain is small in comparison with the distance, one can write for its field $\bar{p}^2 \sim \frac{W}{r^2}$, wherein W is the summative radiation of the chain; it is obviously proportioned to the length of the radiating part of the chain, that is $\sqrt{r}$.

As a result we have $\bar{p}^2 \sim \frac{\sqrt{r}}{r^2} = \frac{1}{r^{3/2}}$, that is, the law of three half. From these considerations follows the average law of attenuation for channels, the transversal dimensions of which are large in comparison with the wave length. In such a case there radiates not a chain but a certain plane region, the linear dimensions of which are $\sim \sqrt{r}$, and the area $\sim r$. Consequently, the total radiation will be $\sim r$ and the law of attenuation $\bar{p}^2 \sim \frac{1}{r}$.)

$$2h_{max} \sim \sqrt{\frac{rh}{p'}};$$

consequently, for the maximum angles we have

$$\alpha_{max} = \frac{h_{max}}{r} \sim \frac{1}{2}\sqrt{\frac{h}{p'r}}.$$

This angle we assume to be small.

Considerations pertaining to case III are presented in an analogous manner.

Section 3. Summary. Limits of Applicability of the Results Obtained

Let us review the above-derived laws of attenuation, with distance, of the acoustic pressure and components of an electromagnetic field. In so doing it is convenient in lieu of taking the usual distance r to utilize the non-dimensional distance ρ , determined for various cases by the formula (20)

(Footnote: Herein we omit from consideration the case of totally reflecting boundaries of the stratum in which the law of attenuation is known to be cylindrical.)

In the case of large non-dimensional distances, i.e., when

$$\rho \gg 1 \qquad (46)$$

of primary importance is the lateral wave, the square of amplitude of which decreases according to the law $\sim \frac{1}{\rho^4}$ (see /1/). On decrease of the distance there "comes into play" the least-damped wave of the discrete spectrum, the square of amplitude of which decreases according to the law $\sim \frac{e^{-\rho}}{\rho}$ (see (26)). At distances $\rho \sim 1$, this wave becomes of primary importance (if n_1 and n_2 are complex, that is if in the media with $z = \pm \infty$ damping takes place, the lateral wave then will be damped exponentially with the distance. ~~(if n_1 and n_2 are complex, that is).~~ If this damping is found to be greater than that of the last wave of the discrete spectrum, the lateral wave then plays no part at any of the distances).

On decrease of the distance there begin to act other waves of the discrete spectrum. The law will become more complicated and is given by formula (15) in which $\sigma(\kappa)$ for the different cases is shown in the drawing, wherein in lieu of r, appears the non-dimensional distance ρ . On further decrease of the distance, when there will prevail, roughly, the condition

$$\rho \ll 1 \qquad (47)$$

(more precisely one should write here $\rho^{s} \ll 1$, wherein for the various cases $\frac{1}{3} \leqslant s \leqslant \frac{3}{2}$. See Section 1) the law of attenuation again acquires a simple form. At these distances there are acting a large number of waves of the discrete spectrum, the sum with respect to which can be replaced by an integral. As a result there are obtained exponential laws of attenuation, of the type of the "three half" law (24).

On further decrease of the distance these laws cease to hold, since there enter into action waves of large numbers ℓ, which do not satisfy conditions (5). From (21) it is apparent that in cases I and II the largest still significant ℓ is equal in order of magnitude to $\ell_{max} \sim \frac{1}{\sqrt{\rho}}$, while in the case III it is $\ell_{max} \sim \frac{1}{\sqrt[3]{\rho}}$.

Thereafter, taking into account also (20), the condition (5) can be written:

case I: $$\left(\rho' \frac{h}{r}\right)^{1/2} \ll 1, \qquad (48)$$

case III: $$\left(\alpha \frac{h}{r}\right)^{2/3} \ll 1, \qquad (49)$$

and an analogous to (48) condition for case II. These conditions are the ~~same~~ ones that determine the lower limit of distances, at which hold the above-derived average laws of attenuation. It is required that distance r be sufficiently large in comparison with the thickness of the stratum h. At shorter distances the average field depends not only on r but also on the positions of the source of radiation and received with respect to the boundaries of the stratum.

Literature Cited

1. Brekhovskikh, L. M. Izvestiya Akademii nauk SSSR, Seriya Fizicheskaya 13, No 5, 515,1949.

2. Brekhovskikh, L. M. Ibid. 10, 491, 1946

DEBATE CONCERNING THE PAPER OF L.M. BREKHOVSKIKH, ON A NEW ME* THOD OF SOLVING PROBLEMS RELATING TO THE FIELD OF A POINT SOURCE OF RADIATION IN A STRATIFIED-INHOMOGENEOUS MEDIUM (THE PAPER WAS PRESENTED AT THE SEPTEMBER 26, 1949 SESSION OF THE DEPARTMENT OF PHYSICO-MATHEMATICAL SCIENCES, ACADEMY OF SCIENCES USSR, AND WAS PUBLISHED IN THE IZVESTIYA AKADEMII NAUK SSSR, SENYA FIZICHESKAYA, 13, NO 4, 409 (1949)

V. N. Kessenikh

You mentioned the absolute identity of the formal aspects in acoustic and electromagnetic eases. Apparently in the electromagnetic case you had in mind only the vertical dipole, since for a horizontal dipole there is obtained not one, but two components of Hertz vector.

L. M. Brekhovskikh

Yes, I agree. However, the general method, which is being presented here, can be transferred without any difficulties to the case of a horizontal dipole.

V. N. Kessenikh

You have not estimated how the picture of lateral waves formation will be altered in cases of a horizontal dipole? You remember the work made known as early as 1912, relative to the rise of directionality in radiation of a horizontal dipole over a semi-conductor surface?

L. M. Brekhovskikh

If the surface is appreciably conductive, the lateral wave is no longer of substantial importance, since it rapidly undergoes damping with distance. Hence in the classic instance of propagation of waves over a semi-conductor surface, the lateral wave can be disregarded. In such a case the lateral wave will be Sommerfeld integral Q_2. The lateral wave can be found to be important if consideration is given to inhomogeneity of the atmosphere.

I have not computed the lateral wave for the horizontal case. One may deem, however, that the general properties of the lateral wave in this case will be the same as for the vertical dipole. Some change may occur only in the function characterizing the dependence of wave amplitude on the vertical coordinate.

V. N. Kessenikh

What about the problem of non-parallel strata, can it be solved? In what direction is the problem to be stated?

L. M. Brekhovskikh

I have not worked on the problem of non-parallel strata. I believe that the method I have proposed will be of use here also.

N. V. Zvolinskiy

The separation into discrete and continuous spectra arrived at in your paper appears from a physical standpoint to be a most enticing result. But I feel somewhat dissatisfied with the analysis presented, which to me appears to be essentially incomplete. Specifically, that integral representation which you have written, will only then become clear, and may be referred to as a presentation of the solution, when you know the singularities of the function within the integration sign. Lacking this, it is not only impossible to move the integration contour, but also one may not even write such an integral. Meanwhile you have omitted the study of poles of the denominator. In your paper you mention very briefly, but even in the theses, which I was in a position to study, the subject is taken up with scarcely more details, and study of the poles there also is incomplete. They contain a number of assumptions, not entirely apparent, and possibly not consonant. So that on final account, we still do not know where the poles are located and how many there are. If a series is written on that basis it is not known whether it converges or not.

Even though the results possess, perhaps, some physical plausibility in general, still their plausibility is doubtful for all falues of parameters and basic functions included in the presentation of the problem. Consequently the region of applicability of your analysis remains uncertain. In my opinion a postulate should still be advanced, with respect to parameters and initial functions, on the basis of which it can be asserted that all this is actually so.

Insufficiency of analysis, it appears to me, affects two problems: first, the study of the denominator poles -- in problems of this kind this is a fundamental, very laborious, but absolutely indispensible process and I fail to see how it can be skipped over; secondly, estimation of the integral by the method of anticline point, a problem which to me is more readily tackled. But even insofar as the last problem is concerned, it also has not been completed. You assume that at the anticline point the first derivative becomes zero. But it can happen that there the second derivative also becomes zero. Such a possibility is not excluded. True, if the problem comprises several parameters such an eventuality is somewhat fortuitous. Still it should be studied.

Therefore I believe that although the results are physically most interesting and attractive, they are not substantiated. You said you were not particularly interested in generality. It seems to me that quite to the contrary: you do not want to consider a concrete medium but consider an arbitrary stratified-inhomogeneous medium. It seems to me that under these conditions even with concretely assigned properties, performance of this study will be unbelievably difficult.

The problem could be presented in a different plane (I refer to our informal conversations) -- that this is not a task of physicists but one of mathematicians. But I think that any physicist utilizing a mathematical method must carry the task to conclusion, the same as any mathematician working on problems capable of application must pursue the task to a stage at which the physical conclusions are clear. Otherwise there is left a feeling of uncertainty, of incompleteness.

A. N. Tikhonov

Solution of problems of mathematical physics is effected in this manner: the first stage is the analytical presentation of a solution, the second stage is interpretation of the analytical expression obtained.

The analytical presentation of the solution has been obtained by you using the method of plane waves. The problem, however, displays cylindrical symmetry. Hence one may separate Bessel function, and write the solution forthwith even in a somewhat general form. The method of plane waves appears to me to have no advantages in the solution of the proposed problem.

So much concerning the analytical for of the solution.

In interpreting the solution, of very great interest is the fact that by the strata of greatest conductivity is transmitted the major portion of energy, and transmission of energy over great distances is attained. In your study of the solution you have segregated two parts: a part corresponding to the discrete spectrum, and a lateral wave representing the solution at a very great distance, when the discrete spectrum is eliminated.

Of greatest interest is the problem of passage of process of one kind into another: in what systems, at what distances the greatest portion of energy is connected with the discrete spectrum, and beginning at what distance this effect is eliminated and the solution is determined by the lateral wave.

Representation of the lateral wave in the form given by you is very interesting, but in addition to this from the standpoint of solution interpretation it is interesting and important to know from what distance on is the solution determined by the lateral wave.

V. N. Kessenikh

First of all I wish to say a few words concerning the reproof directed at the Speaker relative to inadequate study of the function within the sign of integration at the region of the summit point.

Since this work is of primary interest to those concerned with problems relating to the field of electromagnetic and sound waves propagation, the solution is quite obvious to this group of people. Behavior of the function within the integration sign at the summit point region is in fact such as to render the summit method applicable to the fullest extent. This summit point is of the same kind as that of the expression within the integration sign, of, let us say, Hankel function.

The second remark is concerned with further development of researches by L. M. Brekhovskikh. I believe that confining oneself merely to mentioning the existence of an analogy between the studied instance of a vertical dipole and that of a horizontal dipole or of a dipole of arbitrary direction, is not sufficient. This is known from the theory of wave propagation, is known if for no other reason than that the theory of diffractional propagation of radio waves around a sphere utilizes up to now a very imperfect substitute for a horizontal electrical dipole, namely a vertical magnetic dipole.

Transition to a horizontal dipole involves loss of symmetry, and transition to the unsymmetrical case requires utilization of a considerably more complex mathematical apparatus.

In the case of plane-stratified medium, this transition has been effected only in part -- for the plane separation surface of two media. The plane case is of great practical value in connection, let us say, the running wave antenna of Shchukin and a number of other instances. In the plane-stratified medium this problem has not been investigated, although in many instances atmospheric waveguides utilization is important alos in the case of horizontal polarization. It would be most desirable and interesting to continue studies in the direction of investigation of instances of horizontal polarization

L. M. Brekhovskikh

I agree that a solution in the form of quadratures can be obtained also without the use of plane waves. This, however, can be done only in those cases when the wave equation is solved in known functions. My solution is formulated by means of the coefficient of reflection of plane waves at the lower and upper boundaries. In a whole series of cases solution of the wave equation cannot be determined. On such an instance, using known methods, no solution can be obtained for the field of a spherical wave. On the other hand my solution containing the vunctions V_1 and V_2 will be suitable for such cases. The results previously given make it possible, for example, to determine the values of the fields in those instances when dependence of parameters of the medium on the coordinate z is given graphically.

i accept the censures concerning the insufficient completeness of discrete spectrum study. This study constitutes a difficult mathematical problem, but of course it must be solved to complete the task.

To write an integral expression for psi is possible even without a study of the discrete spectrum, since convergence of the integrals has been demonstrated. In this respect I do not agree with N. V. Zvolinskiy. I have made a study, moreover, of the pole behavior in remote regions and regions close to origin of the coordinates, as well as evaluated the distance at which the lateral wave begins to prevail over the discrete spectrum.

I believe that the results obtained permit to operate with certainty in concrete instances. Specifically, I have obtained concrete characteristics of propagation of the first wave of the discrete spectrum which is propagated over the greatest distances, as well of some of the succeeding waves. It seems to me that from a